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A Resolvent Approach to the Real Quantum Plane

VASYL OSTROVSKYI, KONRAD SCHMÜDGEN

Abstract. Let $q \neq \pm 1$ be a complex number of modulus one. This paper deals with the operator relation $AB = qBA$ for self-adjoint operators A and B on a Hilbert space. Two classes of well-behaved representations of this relation are studied in detail and characterized by resolvent equations.

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1. Introduction

The algebraic relation $ab = qba$ is a basic ingredient of the theory of quantum groups. Let us assume for a moment that this relation holds for a complex number q and some elements a and b of a unital $*$-algebra with involution $x \to x^+$. There are three important cases in which this relation is invariant under the involution. The first one is when a is unitary (that is, $a^+a = aa^+ = 1$) and b is hermitian (that is, $b^+ = b$), while in the second case we have $a = b^+$. In both cases q is real. From an operator-theoretic point of view these two cases are closely related (for instance, by taking the polar decomposition of a in the second case). In the third case a and b are hermitian and q is of modulus one. All three cases occur in the definitions of real forms of quantum groups and quantum algebras, see e.g. [7, Subsections 6.1.7, 9.2.4, 9.2.5]. The present paper deals with operator representations of the relation $ab = qba$ in this third case. The corresponding $*$-algebra generated by a and b is the coordinate algebra of the real quantum plane [13] and of the quantum $ax + b$-group [14].

The general operator relation $ab = qba$ has been studied in many papers such as [9], [2], [11], [12], [10], [15], [6], [3].

Throughout this paper q is a fixed complex number of *modulus one* such that $q^2 \neq 1$ and A and B are *self-adjoint* operators on a Hilbert space $\mathcal{H}$. We write

$$q = e^{-\mathrm{i}\theta_0}, \quad \text{where} \quad 0 < |\theta_0| < \pi. \tag{1}$$

Our aim is to study the operator relation

$$AB = qBA. \tag{2}$$

It turns out that this simple operator relation leads to unexpected technical difficulties and interesting operator-theoretic phenomena. If A and B are bounded and $AB = 0$, then (2) is obviously satisfied. Let us call representations of (2) with $AB = 0$ trivial. Since $q^2 \neq 1$, these are the only representations of (2) given by *bounded* operators (see [2] or [9]). Operator representations of algebraic relations have been extensively studied in [9], but the methods developed therein lead only to trivial representations of (2). Further, as noted in [11, p. 1031], in contrast to Lie algebra relations the method of analytic vectors fails for the relation (2).

Representations of (2) by *unbounded* self-adjoint operators A and B have been investigated in [11] and [12]. Some classes of well-behaved representations of (2) have been introduced and classified in [12]. The present paper is devoted to an approach to the operator relation (2) that is based on the resolvents of the self-adjoint operators A and B. For two classes $\mathcal{C}_0$ and $\mathcal{C}_1$ (see Definition 2) of well-behaved representations this approach is developed in detail.

This paper is organized as follows. In Section 2 we give a number of reformulations of the operator relation (2) in terms of the resolvent $R_\lambda(A)$ and B, the resolvent $R_\mu(B)$ and A, and the resolvents $R_\lambda(A)$ and $R_\mu(B)$, and we study the largest linear subspace $\mathcal{D}_q(A,B)$ on which relation (2) holds. In Section 4 the two classes $\mathcal{C}_0$ and $\mathcal{C}_1$ of well-behaved representations of relation (2) are defined and investigated in detail. All irreducible pairs of these classes are built of self-adjoint operators $e^{\alpha Q}$ and $e^{\beta P}$ on the Hilbert space $L^2(\mathbb{R})$, where $Q = x$, $P = \mathrm{i}\frac{d}{dx}$ and $\alpha, \beta \in \mathbb{R}$. We prove that the weak resolvent forms $qR_{\lambda q}(A)B \subseteq BR_\lambda(A)$ and $R_\mu(B)A \subseteq qAR_{\mu q}(B)$ of relation (2) hold for all pairs $\{A,B\}$ of these classes and for all complex numbers λ resp. μ outside certain critical sectors. Section 5 contains the main results of this paper. These are various theorems which characterize (under additional technical assumptions) well-behaved representations, especially pairs $\{A,B\}$ of the classes $\mathcal{C}_0$ and $\mathcal{C}_1$, by weak resolvent relations such as $qR_{\lambda q}(A)B \subseteq BR_\lambda(A)$.

Let $A := e^{\alpha Q}$, $B := e^{\beta P}$, and $q := e^{-\mathrm{i}\alpha\beta}$, where $\alpha, \beta \in \mathbb{R}$. As shown in Section 4, the resolvent relations (12) and (13) are satisfied on $L^2(\mathbb{R})$ if $|\alpha\beta| < \pi$ and λ, μ are not in the critical sector $\mathcal{S}(q)^+$. From Propositions 1 and 2 it follows that for arbitary numbers $\alpha, \beta, \lambda, \mu$ the relations (12) and (13) holds for vectors of the closures of subspaces $(B-\mu I)(A-\lambda I)\mathcal{D}_0$ and $(A-\lambda qI)(B-\mu qI)\mathcal{D}_0$, respectively, where $\mathcal{D}_0 = \mathrm{Lin}\,\{e^{-\varepsilon x^2+\gamma x}; \varepsilon > 0, \gamma \in \mathbb{C}\}$. In Section 6 the orthogonal complements of these two subspaces are explicitely described and the resolvent actions on these complements are computed.

Some technical preliminaries are contained in Section 3. Amongs these are properties of the operator $e^{\beta P}$ and a formula for fractional powers of sectorial operators.

Let us collect some basic notations on operators. Let T be a densely defined closed operator on a Hilbert space. We denote its domain by $\mathcal{D}(T)$, its resolvent set by $\rho(T)$ and its resolvent $(T-\lambda I)^{-1}$ by $R_\lambda(T)$. Let U_T be the phase operator occuring in the polar decomposition $T = U_T|T|$ of the operator T. The symbol $L^2(\mathbb{R})$ stands for the L^2-space with respect to the Lebesgue measure on $\mathbb{R}$.

2. General considerations on the relation (2)

The following two propositions contain some simple reformulations of equation (2) in terms of the resolvents of the self-adjoint operators A and B.

Proposition 1. *Suppose that λ, $\lambda q \in \rho(A)$ and μ, $\mu q \in \rho(B)$.*

(*i*) *If $\mathcal{D}$ is a linear subspace of $\mathcal{D}(AB) \cap \mathcal{D}(BA)$ and* (2) *holds for all $f \in \mathcal{D}$, then*

$$BR_\lambda(A)g = qR_{\lambda q}(A)Bg \tag{3}$$

for all $g \in \mathcal{E} := (A-\lambda I)\mathcal{D}$ and $\mathcal{E}$ is a linear subspace of $\mathcal{D}(B)$.

(*ii*) *If $\mathcal{E}$ is a linear subspace of $\mathcal{D}(B)$ such that $R_\lambda(A)g \in \mathcal{D}(B)$ and* (3) *is satisfied for all $g \in \mathcal{E}$, then* (2) *holds for all $f \in \mathcal{D} := R_\lambda(A)\mathcal{E}$.*

(iii) If $\mathcal{E}$ is a linear subspace of $\mathcal{D}(B)$ and (3) *holds for all $g \in \mathcal{E}$, then*

$$R_\lambda(A)R_\mu(B)h = qR_{\mu q}(B)R_{\lambda q}(A)h + \mu\lambda q(q-1)R_{\mu q}(B)R_{\lambda q}(A)R_\lambda(A)R_\mu(B)h \tag{4}$$

for all $h \in \mathcal{F} := (B - \mu I)\mathcal{E}$.

(iv) If $\mathcal{F}$ is a linear subspace of $\mathcal{H}$ such that (4) *holds for all $h \in \mathcal{F}$, then* (3) *is fulfilled for all $g \in \mathcal{E} := R_\mu(B)\mathcal{F}$.*

Proof. (i): Clearly, (2) implies that

$$(A - \lambda qI)Bg = qB(A - \lambda I)g$$

for $f \in \mathcal{D}$. Hence, for all vectors of the form $g = (A - \lambda I)f$, where $f \in \mathcal{D}$, we have $R_\lambda(A)g \in \mathcal{D}(B)$ and

$$R_{\lambda q}(A)(A - \lambda qI)BR_\lambda(A)g = qR_{\lambda q}(A)B(A - \lambda I)R_\lambda(A)g,$$

so that

$$BR_\lambda(A)g = qR_{\lambda q}(A)Bg.$$

(iii): Let $g \in \mathcal{E}$. From equation (3) we obtain

$$\begin{aligned}(B - \mu qI)R_\lambda(A)g &= (BR_\lambda(A) - \mu qR_\lambda(A))g = (qR_{\lambda q}(A)B - \mu qR_\lambda(A))g\\ &= (qR_{\lambda q}(A)(B - \mu I) + \mu qR_{\lambda q}(A) - \mu qR_\lambda(A))g\\ &= (qR_{\lambda q}(A)(B - \mu I) + \mu q(\lambda q - \lambda)R_{\lambda q}(A)R_\lambda(A))g. \end{aligned} \tag{5}$$

Setting $h = (B - \mu I)g$, we have $g = R_\mu(B)h$. Inserting this into (5) and applying $R_{\mu q}(B)$ to both sides yields (4) for $h \in (B - \mu I)\mathcal{E}$.

(ii) and (iv) follow by reversing the preceding arguments of proofs of (i) and (iii), respectively. □

Using the equalities $R_\lambda(\overline{q}A) = qR_{\lambda q}(A)$ and $R_\mu(\overline{q}B) = qR_{\mu q}(B)$ one can rewrite (3) in the form

$$BR_\lambda(A)g = R_\lambda(\overline{q}A)Bg$$

and (4) as

$$qR_\lambda(A)R_\mu(B)h = R_\mu(\overline{q}B)R_\lambda(\overline{q}A)h + \mu R_\mu(\overline{q}B)(\overline{q}R_\lambda(\overline{q}A) - R_\lambda(A))R_\mu(B)h.$$

In a similar manner the following proposition is derived.

Proposition 2. *Suppose that λ, $\lambda q \in \rho(A)$ and μ, $\mu q \in \rho(B)$.*

(i) If equation (2) *is satisfied for all f of a linear subspace $\mathcal{D} \subseteq \mathcal{D}(AB) \cap \mathcal{D}(BA)$, then*

$$R_\mu(B)Ag = qAR_{\mu q}(B)g \tag{6}$$

for all $g \in \mathcal{E} := (B - \mu qI)\mathcal{D}$ and $\mathcal{E}$ is a subspace of $\mathcal{D}(A)$.

(ii) If $\mathcal{E}$ is a linear subspace of $\mathcal{D}(A)$ such that $R_{\mu q}(B)g \in \mathcal{D}(A)$ and (6) *holds for all $g \in \mathcal{E}$, then* (2) *is true for all $f \in \mathcal{D} := R_{\mu q}(B)\mathcal{E}$.*

(iii) If $\mathcal{E}$ is a linear subspace of $\mathcal{D}(A)$ and (6) *is satisfied for all $g \in \mathcal{E}$, then*

$$R_\lambda(A)R_\mu(B)h = qR_{\mu q}(B)R_{\lambda q}(A)h + \mu\lambda q(q-1)R_\lambda(A)R_\mu(B)R_{\mu q}(B)R_{\lambda q}(A)h \tag{7}$$

for all $h \in \mathcal{F} := (A - \lambda qI)\mathcal{E}$.

(iv) If equation (7) *holds for all h of a linear subspace $\mathcal{F} \subseteq \mathcal{H}$, then* (6) *is satisfied for all $g \in \mathcal{E} := R_{\lambda q}(A)\mathcal{F}$.*

Comparing Propositions 1 and 2, especially formulas (4) and (7), we obtain

Corollary 3. *Let λ, $\lambda q \in \rho(A)$ and μ, $\mu q \in \rho(B)$. If equation (2) holds on a linear subspace $\mathcal{D} \subseteq \mathcal{D}(AB) \cap \mathcal{D}(BA)$, then the operators $R_\lambda(A)R_\mu(B)$ and $R_{\mu q}(B)R_{\lambda q}(A)$ commute on the linear space $(A - \lambda qI)(B - q\mu I)\mathcal{D} \cap (B - \mu I)(A - \lambda I)\mathcal{D}$.*

Without further assumptions the linear subspace $(A - \lambda I)\mathcal{D}$ of $\mathcal{D}(B)$ is neither a core for B nor the subspace $(B - \mu I)\mathcal{E}$ is dense in $\mathcal{H}$. Note that (2) for all $f \in \mathcal{D}$ implies that (4) holds for all vectors $h \in (B - \mu I)(A - \lambda I)\mathcal{D}$ and (7) is valid for $h \in (A - \lambda qI)(B - \mu qI)\mathcal{D}$.

Definition 1. $\mathcal{D}_q(A, B) := \{f \in \mathcal{D}(BA) \cap \mathcal{D}(AB) : \; ABf = qBAf\}$.

Obviously, $\mathcal{D}_q(A, B)$ is the largest linear subspace of $\mathcal{H}$ on which relation (2) holds. Of course, for arbitary self-adjoint operators A and B it may happen $\mathcal{D}_q(A, B) = \{0\}$. From Proposition 1 we immediately obtain the following descriptions of the space $\mathcal{D}_q(A, B)$:

$$\begin{aligned}\mathcal{D}_q(A,B) &= R_\lambda(A)\{g \in \mathcal{D}(B) : R_\lambda(A)g \in \mathcal{D}(B) \text{ and } BR_\lambda(A)g = qR_{\lambda q}(A)Bg\}\\ &= R_\lambda(A)R_\mu(B)\{h \in \mathcal{H} : R_\lambda(A)R_\mu(B)h = qR_{\mu q}(B)R_{\lambda q}(A)h\\ &\qquad + \mu\lambda q(q-1)R_{\mu q}(B)R_{\lambda q}(A)R_\lambda(A)R_\mu(B)h\}.\end{aligned}$$

Similarly, Proposition 2 leads to the following descriptions of $\mathcal{D}_q(A, B)$:

$$\begin{aligned}\mathcal{D}_q(A,B) &= R_{\mu q}(B)\{g \in \mathcal{D}(A) : R_{\mu q}(B)g \in \mathcal{D}(A) \text{ and } R_\mu(B)Ag = qAR_{\mu q}(B)g\}\\ &= R_\lambda(A)R_\mu(B)\big\{h \in \mathcal{H} : R_\lambda(A)R_\mu(B)h = qR_{\mu q}(B)R_{\lambda q}(A)h\\ &\qquad + \mu\lambda q(q-1)(A)R_\lambda(A)R_\mu(B)R_{\mu q}(B)R_{\lambda q}h\big\}.\end{aligned}$$

In particular, we have

$$\mathcal{D}_q(A,B) \subset R_\lambda(A)\mathcal{D}(B) \cap R_{\mu q}(B)\mathcal{D}(A).$$

The operator relation (2) is obviously equivalent to the the relation

$$BAf = \overline{q}ABf. \tag{8}$$

Hence $\mathcal{D}_q(A, B) = \mathcal{D}_{\overline{q}}(B, A)$.

If equation (3) holds for all vectors g of the whole domain $\mathcal{D}(B)$, that is, if

$$qR_{\lambda q}(A)B \subseteq BR_\lambda(A), \tag{9}$$

we shall say that relation (9) is the **weak A-resolvent form** of equation (2) for λ, $\lambda q \in \rho(A)$. If equation (6) holds for all vectors g of the domain $\mathcal{D}(B)$, that is, if

$$R_\mu(B)A \subseteq qAR_{\mu q}(B), \tag{10}$$

we say that relation (10) is the **weak B-resolvent form** of equation (2) for μ, $\mu q \in \rho(B)$. Setting $\nu = \mu q$ relation (10) can be rewritten as

$$\overline{q}R_{\nu\overline{q}}(B)A \subseteq AR_\nu(B). \tag{11}$$

The form (11) of the weak B-resolvent relation of (2) corresponds to the weak A-resolvent form of equation (8) which is obtained by interchanging A and B and replacing q by $\overline{q}$.

Further, if equation (4) is satisfied for all $h \in \mathcal{H}$, that is, if

$$R_\lambda(A)R_\mu(B) = qR_{\mu q}(B)R_{\lambda q}(A) + \mu\lambda q(q-1)R_{\mu q}(B)R_{\lambda q}(A)R_\lambda(A)R_\mu(B), \tag{12}$$

then equation (4) is called the (A, B)-**resolvent form** of equation (2) for λ, $\lambda q \in \rho(A)$ and μ, $\mu q \in \rho(B)$. Likewise, if equation (7) holds for all $h \in \mathcal{H}$, that is, if

$$R_\lambda(A)R_\mu(B) = qR_{\mu q}(B)R_{\lambda q}(A) + \mu\lambda q(q-1)R_\lambda(A)R_\mu(B)R_{\mu q}(B)R_{\lambda q}(A), \tag{13}$$

then equation (7) is called the (B, A)-**resolvent form** of equation (2) for λ, $\lambda q \in \rho(A)$ and μ, $\mu q \in \rho(B)$.

The resolvent relations (12) and (13) can be rewritten as

$$\Big(R_{\mu q}(B)R_{\lambda q}(A) - \frac{1}{\mu\lambda q(q-1)}I\Big)\Big(R_\lambda(A)R_\mu(B) + \frac{1}{\mu\lambda(q-1)}I\Big) = -\frac{1}{\lambda^2\mu^2 q(q-1)^2}I,$$

$$\Big(R_\lambda(A)R_\mu(B) + \frac{1}{\mu\lambda(q-1)}I\Big)\Big(R_{\mu q}(B)R_{\lambda q}(A) - \frac{1}{\mu\lambda q(q-1)}I\Big) = -\frac{1}{\lambda^2\mu^2 q(q-1)^2}I,$$

respectively. They hold for all vectors from the subspaces $(B-\mu I)(A-\lambda I)\mathcal{D}_q(A, B)$ and $(A-\lambda q I)(B-\mu q I)\mathcal{D}_q(A, B)$, respectively. In Section 6 we derive for a class of representations of (2) the form of resolvent relations on the complements of these subspaces.

Proposition 4. *The weak A-resolvent form* (9) *is equivalent to the* (A, B)*-resolvent form* (12) *of equation* (2)*. The weak B-resolvent form* (10) *and the* (B, A)*-resolvent form* (13) *of* (2) *are equivalent.*

Proof. First suppose that (9) holds. This means that (3) is satisfied for all vectors $g \in \mathcal{D}(B)$. Therefore, by Proposition 1(ii), equation (3) holds for $h \in (B-\mu I)\mathcal{D}(B)$. Since $\mu \in \rho(B)$, $(B-\mu I)\mathcal{D}(B)$ is equal to $\mathcal{H}$ which yields (12).

Conversely, assume that (12) is fulfilled. Let $g \in \mathcal{D}(B)$. We set $h = (B - \mu\overline{q}I)g$ in (4). Since the ranges of resolvents of B are contained in the domain of B, the vector in (4) is in $\mathcal{D}(B)$, so we can apply the operator $B - \mu I$ to both sides of (4). Then we obtain (3) which proves (9).

The equivalence of (13) and (10) follows by a similar reasoning. □

The next proposition collects a number of basic facts concerning the weak resolvent identities.

Proposition 5. *Suppose that* $\lambda, \lambda q \in \rho(A)$ *and* $\mu, \mu q \in \rho(B)$.
(i) $qR_{\lambda q}(A)B \subseteq BR_\lambda(A)$ *if and only if* $\mathcal{D}_q(A, B) = R_\lambda(A)\mathcal{D}(B)$.
(ii) $R_\mu(B)A \subseteq qAR_{\mu q}(B)$ *if and only if* $\mathcal{D}_q(A, B) = R_{\mu q}(B)\mathcal{D}(A)$.
(iii) $qR_{\lambda q}(A)B \subseteq BR_\lambda(A)$ *if and only if* $qR_{\overline{\lambda}}(A)B \subseteq BR_{\overline{\lambda q}}(A)$.
(iv) $R_\mu(B)A \subseteq qAR_{\mu q}(B)$ *if and only if* $R_{\overline{\mu q}}(B)A \subseteq qAR_{\overline{\mu}}(B)$.
(v) If $qR_{\lambda q}(A)B \subseteq BR_\lambda(A)$*, then* $\mathcal{D}_q(A, B)$ *is a core for A.*
(vi) If $R_\mu(B)A \subseteq qAR_{\mu q}(B)$ *then* $\mathcal{D}_q(A, B)$ *is a core for B.*

Proof. We carry out the proofs of (i), (iii), and (v). The proofs of (ii), (iv), and (vi) follows by a similar reasoning.

(i): Throughout this proof let us set $\mathcal{D}_B := R_\lambda(A)\mathcal{D}(B)$.

First suppose that $qR_\lambda(A)B \subseteq BR_\lambda(A)$. Obviously, $\mathcal{D}_B \subset \mathcal{D}(A)$. The inclusion $\mathcal{D}_B \subset \mathcal{D}(B)$ follows from $qR_\lambda(A)Bf = BR_\lambda(A)f$, $f \in \mathcal{D}(B)$. Further, we have $B\mathcal{D}_B \subset \mathcal{D}(A)$ since $qR_\lambda(A)Bf = BR_\lambda(A)f$ and $R_\lambda(A)Bf \subseteq \mathcal{D}(A)$, $f \in \mathcal{D}(B)$. Also, $A\mathcal{D}_B \subset \mathcal{D}(B)$, since $A-\lambda I$ maps $\mathcal{D}_B$ onto $\mathcal{D}(B)$. Therefore, $\mathcal{D}_B \subset \mathcal{D}_q(A, B)$. Since $g = (A-\lambda I)f \in \mathcal{D}(B)$ for any $f \in \mathcal{D}_q(A, B)$, we see that $f = R_\lambda(A)g$, so that $\mathcal{D}_q(A, B) \subseteq \mathcal{D}_B$. Thus, $\mathcal{D}_B = \mathcal{D}_q(A, B)$.

Conversely, assume that $\mathcal{D}_B = \mathcal{D}_q(A, B)$. Then $(A-\lambda I)\mathcal{D}_q(A, B) = \mathcal{D}(B)$ and by Proposition 1(i), we have $qR_\lambda(A)Bf = BR_\lambda(A)f$ for all $f \in \mathcal{D}(B)$.

(iii): Suppose that $qR_{\lambda q}(A)B \subseteq BR_\lambda(A)$. Since $R_{\lambda q}(A)$ is bounded, we have $(R_{\lambda q}(A)B)^* = B^*(R_{\lambda q}(A))^* = BR_{\overline{\lambda q}}(A)$ and hence

$$\overline{q}\, BR_{\overline{\lambda q}}(A) = (qR_{\lambda q}(A)B)^* \supseteq (BR_\lambda(A))^* \supseteq R_{\overline{\lambda}}(A)B,$$

so that $qR_{\overline{\lambda}}(A)B \subseteq BR_{\overline{\lambda q}}(A)$.

The converse direction follows by applying the same implication once again.

(v): Since $\mathcal{D}_q(A,B) = R_\lambda(A)\mathcal{D}(B)$ by (i), $(A-\lambda I)\mathcal{D}_q(A,B) = \mathcal{D}(B)$ is dense in $\mathcal{H}$. Hence $\mathcal{D}_q(A,B)$ is a core for A. □

An immediate consequence of Proposition 5 is the following corollary.

Corollary 6. *Let $\lambda, \lambda q \in \rho(A)$ and $\mu, \mu q \in \rho(B)$. Assume that $qR_\lambda(A)B \subseteq BR_\lambda(A)$ and $R_\mu(B)A \subseteq qAR_{\mu q}(B)$. Then*

$$\mathcal{D}_q(A,B) = R_\lambda(A)\mathcal{D}(B) = R_{\mu q}(B)\mathcal{D}(A) \tag{14}$$

and $\mathcal{D}_q(A,B)$ is a core for A and B.

Corollary 7. *Suppose that $\mu, \mu q, \mu q^2 \in \rho(B)$. If $R_\mu(B)A \subseteq qAR_{\mu q}(B)$ and $R_{\mu q}(B)A \subseteq qAR_{\mu q^2}(B)$, then $\mathcal{D}_{q^2}(A^2,B)$ is a core for B.*

Proof. From the assumptions we derive $R_\mu(B)A^2 \subseteq qAR_{\mu q}(B)A \subseteq q^2A^2R_{\mu q^2}(B)$, that is, the weak B-resolvent form for the relation $A^2B = q^2BA^2$ is satisfied. Therefore, $\mathcal{D}_{q^2}(A^2,B)$ is a core for B by Proposition 5(vi). □

The next proposition shows how the resolvent relations (12) and (13) follow from the essential self-adjointness of a certain symmetric operator.

Let us fix a, $b \in \mathbb{R}$ and choose the branch of the square root such that

$$\overline{q}^{1/2} = \overline{q^{1/2}}\,. \tag{15}$$

We define an operator T with domain $\mathcal{D}(T) := \mathcal{D}_q(A,B)$ by

$$Tf = \bar{q}^{1/2}(A - aq^{1/2})(B - bq^{1/2})f + \frac{\bar{q}^{1/2} - q^{1/2}}{2}\, abf, \;\; f \in \mathcal{D}(T). \tag{16}$$

Lemma 8. *The operator T is symmetric.*

Proof. Clearly, $Tf = \big(\bar{q}^{1/2}AB - bA - aB + \frac{q^{1/2}+\bar{q}^{1/2}}{2}\, ab\big)f$. Using this formula we derive

$$\begin{aligned}\langle Tf, g\rangle &= \langle \big(\bar{q}^{1/2}AB - bA - aB + \frac{q^{1/2}+\bar{q}^{1/2}}{2}\, ab\big)f, g\rangle \\ &= \langle f, \big(q^{1/2}BA - bA - aB + \frac{q^{1/2}+\bar{q}^{1/2}}{2}\, ab\big)g\rangle \\ &= \langle f, \big(\bar{q}^{1/2}AB - bA - aB + \frac{q^{1/2}+\bar{q}^{1/2}}{2}\, ab\big)g\rangle = \langle Tf, g\rangle\end{aligned}$$

for $f, g \in \mathcal{D}(T)$, that is, T is symmetric. □

Proposition 9. *Assume that $ab \neq 0$ and $q^2 \neq 1$. If the operator T is essentially self-adjoint, then both resolvent relations (12) and (13) hold on $\mathcal{H}$ for $\lambda = a\bar{q}^{1/2}$, $\mu = b\bar{q}^{1/2}$ and the operator $R_{bq^{1/2}}(B)R_{aq^{1/2}}(A)$ is normal.*

Proof. Setting

$$\tau = \frac{\bar{q}^{1/2} - q^{1/2}}{2}\, ab,$$

the operator T can be rewritten as

$$Tf = \bar{q}^{1/2}(A - aq^{1/2})(B - bq^{1/2})f + \tau f = q^{1/2}(B - b\bar{q}^{1/2})(A - a\bar{q}^{1/2})f - \tau f$$

for $f \in \mathcal{D}(T)$. Therefore, since T is essentially self-adjoint and τ is purely imaginary and nonzero (by the assumptions $ab \neq 0$ and $q^2 \neq 1$), the set

$$\mathcal{F}_0 := (T - \tau I)\mathcal{D}(T) = (A - aq^{1/2})(B - bq^{1/2})\mathcal{D}(T)$$

is dense in $\mathcal{H}$. By Proposition 2,(i) and (iii), equation (7) is satisfied for $\lambda = a\bar{q}^{1/2}$, $\mu = b\bar{q}^{1/2}$ and all vectors $h \in \mathcal{F}_0$. Since $\mathcal{F}_0$ is dense and all resolvent operators are bounded, equation (7) holds for all $h \in \mathcal{H}$. That is, we have

$$\begin{aligned} R_{a\bar{q}^{1/2}}(A)R_{b\bar{q}^{1/2}}(B) &= qR_{bq^{1/2}}(B)R_{aq^{1/2}}(A) \\ &\quad + ab(q-1)R_{a\bar{q}^{1/2}}(A)R_{b\bar{q}^{1/2}}(B)R_{bq^{1/2}}(B)R_{aq^{1/2}}(A). \end{aligned} \tag{17}$$

Thus, the (B, A)-resolvent relation (13) is satisfied.

Similarly, we conclude that

$$\mathcal{F}_1 := (T + \tau I)\mathcal{D}(T) = (B - b\bar{q}^{1/2})(A - a\bar{q}^{1/2})\mathcal{D}(T)$$

is dense in H and equation (4) holds for $\lambda = a\bar{q}^{1/2}$, $\mu = b\bar{q}^{1/2}$ and $h \in \mathcal{F}_1$ by Proposition 1,(i) and (iii), and hence for all vectors $h \in \mathcal{H}$. That is, the (A, B)-resolvent relation (12) is valid and we have

$$\begin{aligned} R_{a\bar{q}^{1/2}}(A)R_{b\bar{q}^{1/2}}(B) &= qR_{bq^{1/2}}(B)R_{aq^{1/2}}(A) \\ &\quad + ab(q-1)R_{bq^{1/2}}(B)R_{aq^{1/2}}(A)R_{a\bar{q}^{1/2}}(A)R_{b\bar{q}^{1/2}}(B). \end{aligned} \tag{18}$$

Comparing (17) and (18) we conclude that

$$R_{a\bar{q}^{1/2}}(A)R_{b\bar{q}^{1/2}}(B)R_{bq^{1/2}}(B)R_{aq^{1/2}}(A) = R_{bq^{1/2}}(B)R_{aq^{1/2}}(A)R_{a\bar{q}^{1/2}}(A)R_{b\bar{q}^{1/2}}(B)$$

which means that the operator $R_{bq^{1/2}}(B)R_{aq^{1/2}}(A)$ is normal. □

3. Operator-theoretic preliminaries

We denote by $P = \mathrm{i}\frac{d}{dx}$ the momentum operator and by $Q = x$ the position operator acting on the Hilbert space $L^2(\mathbb{R})$ with respect to the Lebesgue measure on $\mathbb{R}$. Fix $\beta > 0$.

Lemma 10. *(i) Suppose that $f(z)$ is a holomorphic function on the strip $\mathcal{I}_\beta := \{z \in \mathbb{C} : 0 < \mathrm{Im} z < \beta\}$ such that*

$$\sup_{0<y<\beta} \int_{-\infty}^{+\infty} |f(x + \mathrm{i}y)|^2 \, dx < \infty. \tag{19}$$

Set $f_y(x) := f(x + \mathrm{i}y)$. Then the limits $f_0 := \lim_{y\downarrow 0} f_y(x)$ and $f_\beta := \lim_{y\uparrow\beta} f_y(x)$ exist in $L^2(\mathbb{R})$ and we have $f_0 \in \mathcal{D}(e^{\beta P})$ and $e^{\beta P} f_0 = f_\beta$.

(ii) For each function $f_0 \in \mathcal{D}(e^{\beta P})$ there exists a unique function f as in (i) such that $f_0 := \lim_{y\downarrow 0} f_y(x)$ in $L^2(\mathbb{R})$ and $e^{\beta P} f_0 = f_\beta$.

Proof. [12, Lemma 1.1]. □

If f is a function as in Lemma 10(i), we write simply $f(x)$ for $f_0(x)$ and $f(x+\mathrm{i}\beta)$ for $f_\beta(x)$. Then the operator $e^{\beta P}$ acts by

$$(e^{\beta P})(x) = f(x+\mathrm{i}\beta),\ \ f\in\mathcal{D}(e^{\beta P}). \tag{20}$$

For a nonzero complex number q we denote by $\mathcal{S}(\mathsf{q})^+$ the closed sector in the plane with opening angle less than π between the positive x-axis and the half-line through the origin and $\overline{\mathsf{q}}$ and set $\mathcal{S}(\mathsf{q}) := \mathcal{S}(\mathsf{q})^+ \cup (-\mathcal{S}(\mathsf{q})^+)$.

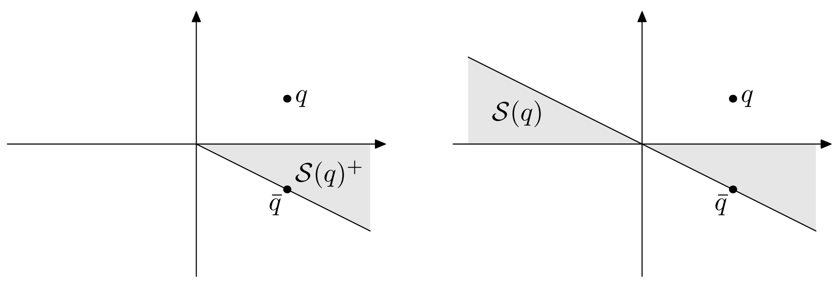

FIGURE 1. The sectors $\mathcal{S}(q)^+$ and $\mathcal{S}(q)$

We fix two reals α,β such that $\beta>0$ and $0<|\alpha\beta|<\pi$. Put $\mathsf{q} := e^{-\mathrm{i}\alpha\beta}$. Now we define positive selfadjoint operators A and B on the Hilbert space $L^2(\mathbb{R})$ by

$$A := e^{\alpha Q} \text{ and } B := e^{\beta P}. \tag{21}$$

Corollary 11. *If $f\in\mathcal{D}(BA)\cap\mathcal{D}(B)$, then $f\in\mathcal{D}(AB)$ and $ABf=\mathsf{q}BAf$.*

Proof. Using the description of the domain $\mathcal{D}(B)=\mathcal{D}(e^{\beta P})$ in Lemma 10 and formula (20) we derive

$$(\mathsf{q}BAf)(x) = qB(e^{\alpha x}f(x)) = e^{-\alpha\mathrm{i}\beta}e^{\alpha(x+\mathrm{i}\beta)}f(x+\mathrm{i}\beta) = e^{\alpha x}f(x+\mathrm{i}\beta) = (ABf)(x).$$

$\square$

Clearly, the linear space

$$\mathcal{D}_0 = \mathrm{Lin}\,\{e^{-\varepsilon x^2+\gamma x};\varepsilon>0,\gamma\in\mathbb{C}\}$$

is contained in $\mathcal{D}(A)\cap\mathcal{D}(B)$ and it is invariant under A and B and also under the Fourier transform and its inverse. By Corollary 11, we have $\mathcal{D}_0\subseteq\mathcal{D}_{\mathsf{q}}(A,B)$. As noted in [12], $\mathcal{D}_0$ is a core for both selfadjoint operators A and B.

Proposition 12. *Suppose that $\lambda\in\mathbb{C}\backslash\mathcal{S}(\mathsf{q})^+$. Then λ and $\lambda\mathsf{q}$ are in $\rho(A)$ and*

$$\mathsf{q}R_{\lambda\mathsf{q}}(A)B \subseteq BR_\lambda(A).$$

Proof. If z runs through the strip $\{z: 0\le \mathrm{Im}\ z\le\beta\}$, then the number $e^{\alpha z}$ fills the sector $\mathcal{S}(\mathsf{q})^+$. Hence the infimum of the function $|e^{\alpha z}-\lambda|$ on the strip $\mathcal{I}_\beta$ is equal to the distance of λ from $\mathcal{S}(\mathsf{q})^+$. In particular, this infimum is positive, since $\lambda\notin\mathcal{S}(\mathsf{q})^+$.

Let $f\in\mathcal{D}(B)$ and let $f(z)$ be the corresponding holomorphic function from Lemma 10. Since $|e^{\alpha z}-\lambda|$ has a positive infimum on the strip $\mathcal{I}_\beta$, the function $g(z)=(e^{\alpha z}-\lambda)^{-1}f(z)$ is holomorphic on $\mathcal{I}_\beta$ and it satisfies condition (19) as well, because f does. Therefore, from Lemma 10 we conclude that $g\in\mathcal{D}(B)$ and

$$\begin{aligned}(BR_\lambda(A)f)(x) &= (Bg)(x) = g(x+\mathrm{i}\beta) = (e^{\alpha(x+\mathrm{i}\beta)}-\lambda)^{-1}f(x+\mathrm{i}\beta)\\ &= \mathsf{q}(e^{\alpha x}-\lambda\mathsf{q})^{-1}f(x+\mathrm{i}\beta) = (\mathsf{q}R_{\lambda\mathsf{q}}(A)Bf)(x).\end{aligned}$$

$\square$

Another technical ingredient used below is Balakrishnan's theory of fractional powers of nonnegative operators on Banach spaces [1], see e.g. [5].

Suppose that T is a closed linear operator on a Banach space such that

$$(-\infty, 0) \subseteq \rho(T) \quad \text{and} \quad \sup\{||\lambda(T+\lambda I)^{-1}|| : \lambda > 0\} < \infty. \tag{22}$$

Then, for any $\gamma \in \mathbb{C}$, $0 < \operatorname{Re}\gamma < 1$, the Balakrishnan operator J^γ (see [5], p. 57) is defined by

$$J^\gamma f = \frac{\sin(\varepsilon + \mathrm{i}t)\pi}{\pi} \int_0^\infty \lambda^{\gamma-1}(T+\lambda I)^{-1}Tf \, d\lambda, \qquad f \in \mathcal{D}(J^\gamma) := \mathcal{D}(T). \tag{23}$$

Here the integral is meant as an improper Riemann integral of a continuous function on $(0, +\infty)$ with values in the underlying Banach space. The operator J^γ (or its closure) is considered as a power of the operator T with exponent γ.

For our investigations the following special case is sufficient.

Proposition 13. *Suppose that A is a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ such that* $\ker A = \{0\}$ *and let $\vartheta \in \mathbb{R}$, $|\vartheta| < \pi$. Let T denote the normal operator $e^{\mathrm{i}\vartheta}A$ in $\mathcal{H}$. Then, for any $0 < \varepsilon < 1$, $t \in \mathbb{R}$ and $f \in \mathcal{D}(T) = \mathcal{D}(A)$ we have*

$$T^{\varepsilon+\mathrm{i}t} f = e^{\mathrm{i}\vartheta\varepsilon}e^{-\vartheta t}A^{\varepsilon+\mathrm{i}t}f = \frac{\sin(\varepsilon+\mathrm{i}t)\pi}{\pi}\int_0^\infty \lambda^{\varepsilon+\mathrm{i}t-1}(T+\lambda I)^{-1}Tf \, d\lambda, \tag{24}$$

where the operators $T^{\varepsilon+\mathrm{i}t}$ and $A^{\varepsilon+\mathrm{i}t}$ are defined by the spectral functional calculus.

Proof. Using that $|\vartheta| < \pi$ and $A \geq 0$ it is easily verified that the operator T satisfies the conditions stated in (22). Hence formula (23) for the Balakrishnan operator $J^{\varepsilon+\mathrm{i}t}$ holds. For the normal operator T the closure of the operator $J^{\varepsilon+\mathrm{i}t}$ is just the power $T^{\varepsilon+\mathrm{i}t}$ defined by the functional calculus (see Example 3.3.2 in [5]), where the principal branch of the complex power has to be taken. Further, since $|\vartheta| < \pi$, we have $T^{\varepsilon+\mathrm{i}t} = e^{\mathrm{i}\vartheta\varepsilon}e^{-\vartheta t}A^{\varepsilon+\mathrm{i}t}$. Hence formula (24) follows from (23). $\square$

Lemma 14. *If A is a positive self-adjoint operator with trivial kernel, then*

$$\lim_{\varepsilon\to+0} A^\varepsilon f = f \quad \text{for} \quad f \in \mathcal{D}(A).$$

Proof. By the spectral calculus we have

$$\|A^\varepsilon f - f\|^2 = \int_0^\infty |\lambda^\varepsilon - 1|^2 \, d\langle E(\lambda)f, f\rangle.$$

Passing to the limit $\varepsilon \to +0$ and using Lebesgue's dominated convergence theorem (by the assumption $f \in \mathcal{D}(A)$) we obtain the assertion. $\square$

4. Two Classes of Well-behaved Representations of Relation (2)

In this section we describe some well-behaved representations of relation (2). For this we also restate some results from [12].

Recall that $q = e^{-\mathrm{i}\theta_0}$ and $0 < |\theta_0| < \pi$ by (1). Set $\theta_1 := \theta_0 - \pi$ if $\theta_0 > 0$, $\theta_1 := \theta_0 + \pi$ if $\theta_0 < 0$. Then we also have

$$-q = e^{-\mathrm{i}\theta_1} \quad \text{and} \quad 0 < |\theta_1| < \pi.$$

If $A = 0$ or if $B = 0$, then $\mathcal{D}_q(A, B) = \mathcal{D}(B)$ resp. $\mathcal{D}_q(A, B) = \mathcal{D}(A)$ and it is obvious that the pair $\{A, B\}$ satisfies the relation (2) and the resolvent relations (3) and (4). We call pairs of the form $\{0, B\}$ and $\{A, 0\}$ *trivial representations* of relation (2).

Interesting representations of relation (2) are the classes $\mathcal{C}_0$ and $\mathcal{C}_1$ defined as follows.

Definition 2. *Suppose that* $\ker A = \ker B = \{0\}$. *We say that the pair* $\{A, B\}$ *is a* representation of the class $\mathcal{C}_0$ *if*

$$|A|^{\mathrm{i}t} B \subseteq e^{\theta_0 t} B|A|^{\mathrm{i}t}\ ,\ t \in \mathbb{R},\ \text{and}\ U_A B \subseteq B U_A. \tag{25}$$

and that the pair $\{A, B\}$ *is a* representation of the class $\mathcal{C}_1$ *if*

$$|A|^{\mathrm{i}t} B \subseteq e^{\theta_1 t} B|A|^{\mathrm{i}t}\ ,\ t \in \mathbb{R},\ \text{and}\ U_A U_B = -U_B U_A. \tag{26}$$

Definition 3. *The trivial pairs* $\{A_2, 0\}$, $\{0, B_2\}$ *and pairs* $\{A_0, B_0\}$ *and* $\{A_1, B_1\}$ *of the classes* $\mathcal{C}_0$ *and* $\mathcal{C}_1$*, respectively, and orthogonal direct sum of such pairs are called* well-behaved representations *of relation (2).*

Remarks. 1. Note that the class $\mathcal{C}_0$ defined above is precisely the class $\mathcal{C}_0$ in [12], while the class $\mathcal{C}_1$ according to Definition 2 corresponds to $\mathcal{C}_1$ if $\theta_0 < 0$ and to $\mathcal{C}_{-1}$ if $\theta_0 > 0$ in [12].

2. Suppose that $\{A, B\}$ is a well-behaved representation of relation (2). If $A \geq 0$ and $\ker A = \ker B = \{0\}$, then $U_A = I$ and $\{A, B\}$ is a pair of the class $\mathcal{C}_0$. Further, if $A \geq 0$, then the well-behaved representation $\{A, B\}$ cannot have an orthogonal summand of the class $\mathcal{C}_1$.

3. As it is usual for relations having unbounded operator representations there are many "bad" unbounded representations of relation (2). In [11] pairs of self-adjoint operators A and B have been constructed for which $\mathcal{D}_q(A, B)$ is a core for A and B, but the pair $\{A, B\}$ is not a well-behaved representation of relation (2) and it is not in one of classes $\mathcal{C}_n$, $n \in \mathbb{Z}$, defined in [12].

Let us describe all pairs of the classes $\mathcal{C}_0$ and $\mathcal{C}_1$ up to unitary equivalence. We fix real numbers $\alpha, \alpha_1, \beta, \beta_1$, where $\beta > 0$, $\beta_1 > 0$, such that

$$\alpha\beta = \theta_0 \quad \text{and} \quad \alpha_1\beta_1 = \theta_1, \quad \text{where} \quad q = e^{-\mathrm{i}\theta_0} \text{ and } \quad -q = e^{-\mathrm{i}\theta_1}. \tag{27}$$

Let $\mathcal{K}$ be a Hilbert space.

Let u, v be two commuting self-adjoint unitaries on $\mathcal{K}$. We define self-adjont operators A and B on the Hilbert space $\mathcal{H} = \mathcal{K} \otimes L^2(\mathbb{R})$ a by

$$A_0 = u \otimes e^{\alpha Q}, \quad B_0 = v \otimes e^{\beta P} \tag{28}$$

and self-adjoint operators A_1 and B_1 on the Hilbert space $\mathcal{H}_1 = (\mathcal{K} \oplus \mathcal{K}) \otimes L^2(\mathbb{R})$ by the operator matrices

$$A_1 = \begin{pmatrix} e^{\alpha_1 Q} & 0 \\ & -e^{\alpha_1 Q} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & e^{\beta_1 P} \\ e^{\beta_1 P} & 0 \end{pmatrix}. \tag{29}$$

Proposition 15. *The pairs* $\{A_0, B_0\}$ *and* $\{A_1, B_1\}$ *belong to the classes* $\mathcal{C}_0$ *and* $\mathcal{C}_1$*, respectively. Each pair of the class* $\mathcal{C}_0$ *resp.* $\mathcal{C}_1$ *is unitarily equivalent to a pair* $\{A_0, B_0\}$ *resp.* $\{A_1, B_1\}$ *of the form (28) resp. (29).*

Corollary 16. *Up to unitary equivalence there are precisely five nontrivial irreducible well-behaved representations of relation (2). These are the fours pairs* $\{A = \varepsilon_1 e^{\alpha Q}, B = \varepsilon_2 e^{\beta P}\}$ *on* $L^2(\mathbb{R})$*, where* $\varepsilon_1, \varepsilon_2 \in \{+1, -1\}$*, and the pair* $\{A, B\}$ *on* $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ *given by (29) with* $\mathcal{K} = \mathbb{C}$.

Any well-behaved representation $\{A, B\}$ *of relation (2) satisfying* $\ker A = \ker B = \{0\}$ *is a direct orthogonal sum of these representations.*

Corollary 17. *Let $\{A, B\}$ be a well-behaved representation of relation (2) for which* $\ker A = \ker B = \{0\}$. *Then there is a linear subspace $\mathcal{D} \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ such that*
(i) $A\mathcal{D} = \mathcal{D}$, $B\mathcal{D} = \mathcal{D}$, and $|A|^{\mathrm{i}t}\mathcal{D} = \mathcal{D}$, $|B|^{\mathrm{i}t}\mathcal{D} = \mathcal{D}$ for $t \in \mathbb{R}$,
(ii) $\mathcal{D}$ is a core for A and B,
(iii) $ABf = BAf$ for $f \in \mathcal{D}$.

Corollary 18. *A pair $\{A, B\}$ is a well-behaved representation (resp. of the class $\mathcal{C}_0$ or $\mathcal{C}_1$) of relation (2) if and only if $\{B, A\}$ is a well-behaved representation (resp. of the class $\mathcal{C}_0$ or $\mathcal{C}_1$) of relation (8).*

Proposition 15 and Corollaries 16–18 are contained in [12, Section 2].

The next proposition is essentially used in the proofs of various theorems in Section 5.

Proposition 19. *Let $k = 0, 1$. Suppose that* $\ker A = \ker B = \{0\}$ *and $\mathcal{D}_q(A, B)$ is a core for B. If*

$$|A|^{\mathrm{i}t} B \subseteq e^{\theta_k t} B |A|^{\mathrm{i}t} \quad \text{for} \quad t \in \mathbb{R}, \tag{30}$$

then $\{A, B\}$ is a pair of the class $\mathcal{C}_k$.

Proof. Putting $q_k := (-1)^k q$ we have $q_k = e^{\mathrm{i}\theta_k}$. By (30), Proposition 2.3 in [12] applies to the pair $\{|A|, B\}$ and the relation $|A|B = q_k B|A|$. Hence there exists a linear subspace $\mathcal{D}$ of $\mathcal{D}_{q_k}(|A|, B)$ such that $\mathcal{D} = |A|\mathcal{D}$ is a core for B. Then $|A|Bg = q_k B|A|g$ for $g \in \mathcal{D}$ by the definition of $\mathcal{D}_{q_k}(|A|, B)$. Since A is self-adjoint and $\ker A = \{0\}$, U_A is self-adjoint unitary and $A = |A|U_A$.

Let $f \in \mathcal{D}_q(A, B)$ and $g \in \mathcal{D}$. Using the preceding facts we derive

$$\begin{aligned} \langle U_A f, B|A|g\rangle &= \langle U_A f, \overline{q_k}|A|Bg\rangle = \langle f, \overline{q_k} U_A |A| Bg\rangle = \langle f, \overline{q_k} ABg\rangle = \langle q_k Af, Bg\rangle \\ &= \langle (-1)^k qBAf, g\rangle = \langle (-1)^k ABf, g\rangle = \langle (-1)^k U_A Bf, |A|g\rangle. \end{aligned}$$

Since $\mathcal{D} = |A|\mathcal{D}$ is a core for B, from the preceding equality we conclude that $U_A f \in \mathcal{D}(B)$ and $BU_A f = (-1)^k U_A Bf$ for $f \in \mathcal{D}_q(A, B)$. By assumption $\mathcal{D}_q(A, B)$ is a core for B, so the latter implies that $U_A B \subseteq (-1)^k BU_A$.

Since U_A is a self-adjoint unitary, we get $U_A B U_A \subseteq (-1)^k B$, that is, the self-adjoint operator $(-1)^k B$ is an extension of the self-adjoint operator $U_A B U_A$ on $\mathcal{H}$. This is only possible if $U_A B U_A = (-1)^k B$. From the latter it follows that $U_A |B| U_A = |B|$ and hence $|B|U_A = U_A U_A |B| U_A = |B| U_A$. Therefore,

$$U_A U_B |B| = U_A B \subseteq (-1)^k BU_A = (-1)^k U_B |B| U_A = (-1)^k U_B U_A |B|.$$

Since $\ker B = \{0\}$, the range of $|B|$ is dense in $\mathcal{H}$, so we get $U_A U_B = (-1)^k U_A U_B$. Thus, $\{A, B\}$ is in $\mathcal{C}_k$. $\square$

Now let us return to the weak resolvent equations

$$qR_{\lambda q}(A) B \subseteq BR_\lambda(A), \tag{31}$$

$$\overline{q} R_{\mu\overline{q}}(B) A \subseteq AR_\mu(B). \tag{32}$$

For the trivial representations $\{0, B\}$ resp. $\{A, 0\}$ they are obviously fulfilled for all $\lambda \neq 0$ resp. $\mu \neq 0$. The classes $\mathcal{C}_0$ and $\mathcal{C}_1$ are treated in the next theorem.

Theorem 20. *(i) If $\{A, B\}$ is a pair of the class $\mathcal{C}_0$ for the relation (2), then (31) and (32) are satisfied for all $\lambda \in \mathbb{C}\backslash\mathcal{S}(q)$ and $\mu \in \mathbb{C}\backslash\mathcal{S}(\overline{q})$. If in addition $A \geq 0$ resp. $B \geq 0$, then (31) resp. and (32) holds for $\lambda \in \mathbb{C}\backslash\mathcal{S}(q)^+$ resp. $\mu \in \mathbb{C}\backslash\mathcal{S}(\overline{q})^+$.*

(ii) If $\{A,B\}$ is a pair of the class $\mathcal{C}_1$ for the relation (2), then (31) and (32) are fulfilled for $\lambda \in \mathbb{C}\backslash\mathcal{S}(-q)$ and $\mu \in \mathbb{C}\backslash\mathcal{S}(-\overline{q})$.

Proof. By Corollary 18 it suffices to prove all assertions for the first relation (31). Clearly, (31) is preserved under orthogonal direct sums. Therefore, by Corollary 16, it is sufficient to prove (31) for the corresponding irreducible representations listed in Corollary 16.

(i): Let $\{A_0 = e^{\alpha Q}, B_0 = e^{\beta P}\}$ and suppose that $\lambda \notin \mathcal{S}(q)^+$. Then, by Proposition 12, relation (31) is valid. Then, obviously, (31) holds also for the pair $\{A_0, -B_0\}$. Since $R_z(-A_0) = -R_{-z}(A_0)$, it follows that (31) is satisfied for the pairs $\{-A_0, \pm B_0\}$ provided that $\lambda \notin -\mathcal{S}(q)^+$.

(ii): We have to show that (31) holds for the pair $\{A_1, B_1\}$ given by (29). Put $A := e^{\alpha_1 Q}$, $B := e^{\alpha_1 P}$ and $\mathsf{q} := -q$. Then, since

$$R_z(A) = \begin{pmatrix} R_z(A) & 0 \\ 0 & R_z(-\mathsf{A}) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, \tag{33}$$

the weak resolvent relation $qR_{\lambda q}(A)B \subseteq BR_\lambda(A)$ reduces to the relations

$$qR_{\lambda q}(A)B \subseteq BR_\lambda(-A), \quad qR_{\lambda q}(-A)B \subseteq BR_\lambda(A).$$

Since $R_z(-A) = -R_{-z}(A)$ and $\mathsf{q} := -q$, the latter equalities are equivalent to

$$\mathsf{q}R_{(-\lambda)\mathsf{q}}(A)B \subseteq BR_{(-\lambda)}(A), \quad \mathsf{q}R_{\lambda\mathsf{q}}(A)B \subseteq BR_\lambda(A).$$

But these relations follow from Proposition 12, now applied to $\mathsf{q} := -q$. □

5. Characterizations of Classes $\mathcal{C}_0$ and $\mathcal{C}_1$ by Weak Resolvent Identities

Recall that A and B always denote self-adjoint operators acting on a Hilbert space $\mathcal{H}$ and that the linear subspace $\mathcal{D}_q(A,B)$ was defined in Definition 1.

Let $\{A,B\}$ be a pair of class $\mathcal{C}_0$. If $\lambda > 0$, then $-\lambda$ is not in the sector $\mathcal{S}(q)^+$, so the weak resolvent identity $qR_{-\lambda q}(A)B \subseteq BR_{-\lambda}(A)$ holds by Theorem 20(i). Similarly, if $\mu \in \mathbb{R}\mathrm{i}$, $\mu \neq 0$, and $|\theta| < \frac{\pi}{2}$, then $\mu \notin \mathcal{S}(q)^+$ and hence $qR_{\mu q}(A)B \subseteq BR_\mu(A)$. The following two theorems state some converses of these assertions.

Theorem 21. *Let A is a positive operator such that* $\ker A = \{0\}$. *Suppose that $0 < |\theta_0| < \pi$ and the domain $\mathcal{D}_q(A,B)$ is a core for B. Assume that*

$$qR_{-\lambda q}(A)B \subseteq BR_{-\lambda}(A) \text{ for } \lambda > 0.$$

Then the pair $\{A,B\}$ is an orthogonal direct sum of a trivial representation $\{A_2, 0\}$ and a pair $\{A_0, B_0\}$ of the class $\mathcal{C}_0$.

Proof. Let $f \in \mathcal{D}_q(A,B)$. Cleary, the positive self-adjoint operator A and the normal operator $\overline{q}A$ (because of $q = e^{-\mathrm{i}\theta_0}$ with $|\theta_0| < \pi$) satisfy the assumptions of Proposition 13. By the definition of $\mathcal{D}_q(A,B)$, the vectors f and Bf are in $\mathcal{D}(A)$, so formula (24) applies to the operator $T = A$ and the vector Bf and also to the operator $T = \overline{q}A$ and the vector f. The assumptions $qR_{-\lambda q}(A)B \subseteq BR_{-\lambda}(A)$ and $ABf = qBAf$ imply that

$$(\overline{q}A + \lambda I)^{-1}(\overline{q}A)Bf = B(A + \lambda I)^{-1}Af.$$

Next we apply Proposition 13 to the operator $T = \overline{q}A$. Since $\overline{q} = e^{\mathrm{i}\theta_0}$ with $|\theta_0| < \pi$, the assumptions of Proposition 13 are fulfilled. Interchanging the closed operator B

and the integral in formula (24) (by considering the integral as a limit of $\mathcal{H}$-valued Riemann sums) we therefore obtain

$$(\overline{q}A)^{\varepsilon+\mathrm{i}t}Bf = BA^{\varepsilon+\mathrm{i}t}f \quad \text{for } f \in \mathcal{D}_q(A,B),\ t \in \mathbb{R}. \tag{34}$$

By the first equality in (24) and the relation $A^{\varepsilon+\mathrm{i}t}f = A^{\varepsilon}A^{\mathrm{i}t}f$ we have

$$(\overline{q}A)^{\varepsilon+\mathrm{i}t}Bf = (e^{\mathrm{i}\theta_0}A)^{\varepsilon+\mathrm{i}t}Bf = e^{\mathrm{i}\varepsilon\theta_0}e^{-\theta_0 t}\,A^{\mathrm{i}t}A^{\varepsilon}Bf,$$

so by (34) we obtain

$$e^{\mathrm{i}\varepsilon\theta_0}e^{-\theta_0 t}\,A^{\mathrm{i}t}A^{\varepsilon}Bf = BA^{\varepsilon}A^{\mathrm{i}t}f. \tag{35}$$

Recall that $f \in \mathcal{D}(A)$ and $A^{\mathrm{i}t}Bf \in \mathcal{D}(A)$ by the assumption $f \in \mathcal{D}_q(A,B)$. Passing to the limit $\varepsilon \to +0$ in (35) by using Lemma 14 and the fact that the operator B is closed it follows that $A^{\mathrm{i}t}Bf = e^{\theta_0 t}B\,A^{\mathrm{i}t}f$ for all $f \in \mathcal{D}_q(A,B)$. Since $\mathcal{D}_q(A,B)$ is a core for B by assumption, we conclude that $A^{\mathrm{i}t}B \subseteq e^{\theta t}BA^{\mathrm{i}t}$ for all $t \in \mathbb{R}$. The latter implies that $A^{\mathrm{i}t}$ leaves the closed subspace $\mathcal{H}_2 := \ker B$ invariant. Hence $\mathcal{H}_2$ is reducing for A and B, so we have $B = 0 \oplus B_0$ and $A = A_2 \oplus A_0$ on $\mathcal{H} = \mathcal{H}_2 \oplus \mathcal{H}_2^{\perp}$ such that $\ker B_0 = \{0\}$ and $(A_0)^{\mathrm{i}t}B_0 \subseteq e^{\theta_0 t}B_0(A_0)^{\mathrm{i}t}$ for $t \in \mathbb{R}$. Since $A \geq 0$ and $\ker A = \{0\}$, the pair $\{A_0, B_0\}$ belongs to $\mathcal{C}_0$. □

Theorem 22. *Suppose that* $0 < |\theta_0| < \frac{\pi}{2}$ *and* $\ker A = \{0\}$. *Assume that* $\mathcal{D}_{q^2}(A^2,B)$ *is a core for* B *and*

$$qR_{\mu\mathrm{i}q}(A)B \subseteq BR_{\mu\mathrm{i}}(A) \quad \text{for } \mu \in \mathbb{R}, \mu \neq 0.$$

Then $\{A,B\}$ *is an orthogonal sum of a trivial representation* $\{A_2,0\}$ *and a pair* $\{A_0,B_0\}$ *such that* $\{|A_0|,B_0\}$ *belongs to the class* $\mathcal{C}_0$. *If in addition* $\mathcal{D}_q(A,B)$ *is a core for* B, *then* $\{A_0,B_0\}$ *is a pair of the class* $\mathcal{C}_0$.

Proof. From the relation $qR_{\mu\mathrm{i}q}(A)B \subseteq BR_{\mu\mathrm{i}}(A)$ it follows that each resolvent $R_{\mu\mathrm{i}}(A)$ and its adjoint $R_{-\mu\mathrm{i}}(A)$ leaves the closed linear subspace $\mathcal{H}_2 := \ker B$ invariant. This implies that $\mathcal{H}_2$ is a reducing subspace for $R_{\mu\mathrm{i}}(A)$ and therefore for the operator A. Obviously, $\mathcal{H}_2$ reduces B. Hence the pair $\{A,B\}$ is an orthogonal sum of a trivial representation $\{A_2,0\}$ and a pair $\{A_0,B_0\}$ such that $\ker B_0 = \{0\}$. For notational simplicity let us assume already that $\ker B = \{0\}$. Our aim is to prove that $\{A,B\}$ is in $\mathcal{C}_0$.

First we recall a simple operator-theoretic fact: If T is a closed operator such that $\nu, -\nu \in \rho(T)$, then $\nu^2 \in \rho(T^2)$ and

$$R_{\nu^2}(T^2) = R_{\nu}(T)R_{-\nu}(T) = \frac{1}{2\nu}(R_{\nu}(T) - R_{-\nu}(T)). \tag{36}$$

Suppose now that $\lambda > 0$. Putting $\mu = \sqrt{\lambda}$, we have $(\mathrm{i}\mu)^2 = -\lambda$. Let $f \in \mathcal{D}(B)$. Using the identity (36) twice, for $\nu = \mu\mathrm{i}q$ and for $\nu = \mu\mathrm{i}$, and the assumptions $qR_{\pm\mu\mathrm{i}q}(A)B \subseteq BR_{\pm\mu\mathrm{i}}(A)$, we obtain

$$\begin{aligned} q^2R_{-\lambda q^2}(A^2)Bf &= \frac{q^2}{2\mu\mathrm{i}q}\,(R_{\mu\mathrm{i}q}(A) - R_{-\mu\mathrm{i}q}(A))Bf \\ &= \frac{1}{2\mu\mathrm{i}}\,B(R_{\mu\mathrm{i}}(A) - R_{-\mu\mathrm{i}}(A))f = BR_{-\lambda}(A^2)f. \end{aligned} \tag{37}$$

Thus, since $q^2 = e^{-2\mathrm{i}\theta_0}$, $|2\theta_0| < \pi$ and $\mathcal{D}_{q^2}(A^2,B)$ is a core for B by assumption, the pair $\{A^2,B\}$ satisfies all assumptions of Theorem 21 for the relation $A^2B = q^2BA^2$. Therefore, by this theorem we have $(A^2)^{\mathrm{i}t}B \subseteq e^{2\theta_0 t}B(A^2)^{\mathrm{i}t}$ for $t \in \mathbb{R}$, so that $|A|^{\mathrm{i}s}B \subseteq e^{\theta_0 s}B|A|^{\mathrm{i}s}$ for $s \in \mathbb{R}$. Hence the pair $\{|A|,B\}$ belongs to $\mathcal{C}_0$ (see e.g.

Remark 2 in Section 4). Further, if in addition $\mathcal{D}_q(A,B)$ is a core for B, it follows from Proposition 19 that $\{A,B\}$ is in the class $\mathcal{C}_0$. $\square$

The next theorem contains a characterization of the class $\mathcal{C}_1$. Recall that for the class $\mathcal{C}_1$ the weak resolvent relation (31) holds for $\lambda \in \mathbb{C}\backslash\mathcal{S}(-q)$ by Theorem 20(ii).

Theorem 23. *Suppose that $0<|\theta_0|<\pi$ and $\ker A = \ker B = \{0\}$. Assume that both domains $\mathcal{D}_{q^2}(A^2,B)$ and $\mathcal{D}_q(A,B)$ are cores for the operator B and that there exists a number $p\in\mathbb{C}\backslash\mathcal{S}(-q)$ such that*

$$qR_{\mu pq}(A)B \subseteq BR_{\mu p}(A) \quad \text{for all} \quad \mu\in\mathbb{R}, \mu\neq 0.$$

Then the pair $\{A,B\}$ belongs to the class $\mathcal{C}_1$.

Proof. Without loss of generality we can choose the number $p\in\mathbb{C}\backslash\mathcal{S}(-q)$ of modulus one and contained in the open sector with angle less than π between the positive x-axis and the half-line through the origin and $\overline{q}$. We modify some arguments that have been used already in the proofs of Theorems 21 and 22.

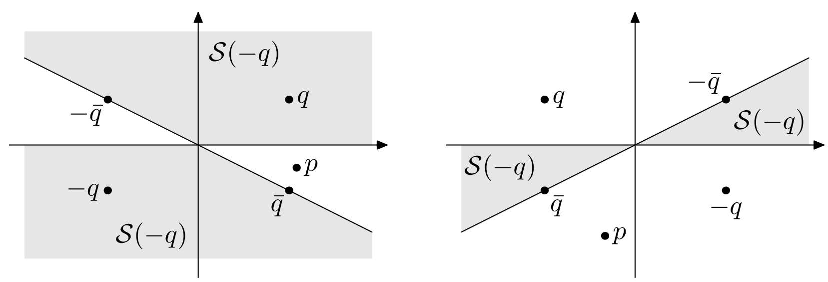

FIGURE 2. The $\mathcal{S}(-q)$ sector and admissible ranges for p as $0<\arg q<\frac{\pi}{2}$ and as $\frac{\pi}{2}<\arg q<\pi$

Using the identity (36) the assumption $qR_{\mu pq}(A)B\subseteq BR_{\mu p}(A)$ implies that

$$q^2R_{\lambda p^2q^2}(A^2)B\subseteq BR_{\lambda p^2}(A^2),\ \lambda>0. \tag{38}$$

Let $f\in\mathcal{D}_{q^2}(A^2,B)$. From (38) and the relation $A^2Bf=q^2BA^2f$ we derive

$$(-\overline{p}^2\overline{q}^2A^2+\lambda I)^{-1}(-\overline{p}^2\overline{q}^2A^2)Bf = B(-\overline{p}^2A^2+\lambda I)^{-1}(-\overline{p}^2A^2)f,\ \lambda>0. \tag{39}$$

Our aim is to apply Lemma 13, especially formula (24) therein, to the operators $-\overline{p}^2\overline{q}^2A^2$ and $-\overline{p}^2A^2$. To fullfill the assumptions of Lemma 13 it is crucial to write the number $-\overline{p}^2\overline{q}^2$ and $-\overline{p}^2$ (of modulus one) in the form $e^{\mathrm{i}\vartheta}$ with $\vartheta\in(-\pi,\pi)$. Let us write p as $p=e^{\mathrm{i}\psi}$ with $0<|\psi|<\pi$ and let $s(\psi)$ denote the sign of ψ. Note that $q=e^{-\mathrm{i}\theta}$ and $|\theta|<\pi$. Also we recall that by definition we have $\theta_1=\theta-\pi$ if $\theta>0$ and $\theta_1=\theta+\pi$ if $\theta<0$. We shall prove that

$$-\overline{p}^2=e^{\mathrm{i}(-2\psi+s(\psi)\pi)} \quad \text{and} \quad -2\psi+s(\psi)\pi\in(-\pi,\pi), \tag{40}$$

$$-\overline{p}^2\overline{q}^2=e^{\mathrm{i}(2\theta_1-2\psi+s(\psi)\pi)} \quad \text{and} \quad 2\theta_1-2\psi+s(\psi)\pi\in(-\pi,\pi). \tag{41}$$

First suppose that $\psi>0$. Then $\pi>\theta>\psi>0$ by the choice of the number p. Hence $-2\psi+\pi\in(-\pi,\pi)$ and $-\overline{p}^2=-(e^{-\mathrm{i}\psi})^2=e^{\mathrm{i}(-2\psi+\pi)}$. Further,

$$2\theta_1-2\psi+\pi=2(\theta-\pi)-2\psi+\pi=2(\theta-\psi)-\pi\in(-\pi,\pi)$$

and $-\overline{p}^2\overline{q}^2=-(e^{-\mathrm{i}\psi})^2(e^{\mathrm{i}\theta})^2=e^{\mathrm{i}(2\theta-2\psi-\pi)}=e^{\mathrm{i}(2\theta_1-2\psi+\pi)}$.

Next we treat the case $\psi < 0$. Then $0 > \psi > \theta > -\pi$ by the definition of p. Therefore, $-2\psi - \pi \in (-\pi, \pi)$ and $-\overline{p}^2 = -(e^{-\mathrm{i}\psi})^2 = e^{\mathrm{i}(-2\psi-\pi)}$. Moreover,

$$2\theta_1 - 2\psi - \pi = 2(\theta + \pi) - 2\psi - \pi = 2(\theta - \psi) + \pi \in (-\pi, \pi)$$

and $-\overline{p}^2\overline{q}^2 = -(e^{-\mathrm{i}\psi})^2(e^{\mathrm{i}\theta})^2 = e^{\mathrm{i}(2\theta-2\psi+\pi)} = e^{\mathrm{i}(2\theta_1-2\psi-\pi)}$. This proves (40) and (41) in both cases.

By (40) and (41) it follows from the first equality of formula (24) that

$$(-\overline{p}^2 A^2)^{\varepsilon+\mathrm{i}t} g = e^{\mathrm{i}(-2\psi+s(\psi)\pi)\varepsilon}\, e^{-(-2\psi+s(\psi)\pi)t}\, (A^2)^{\varepsilon+\mathrm{i}t} g, \tag{42}$$

$$(-\overline{p}^2\overline{q}^2 A^2)^{\varepsilon+\mathrm{i}t} g = e^{\mathrm{i}(2\theta_1-2\psi+s(\psi)\pi)\varepsilon}\, e^{-(2\theta_1-2\psi+s(\psi)\pi)t}\, (A^2)^{\varepsilon+\mathrm{i}t} g \tag{43}$$

for any $g \in \mathcal{D}(A^2)$. Since $f, Bf \in \mathcal{D}(A^2)$, the second equality of formula (24) yields

$$(-\overline{p}^2\overline{q}^2 A^2)^{\varepsilon+\mathrm{i}t} Bf = B(-\overline{p}^2 A^2)^{\varepsilon+\mathrm{i}t} f. \tag{44}$$

Inserting (42) with $g = f$ and (43) with $g = Bf$ into (44) and passing to the limit $\varepsilon \to +0$ we obtain

$$e^{-2\theta_1 t}(A^2)^{\mathrm{i}t} Bf = B(A^2)^{\mathrm{i}t} f,\ t \in \mathbb{R},$$

and hence $|A|^{\mathrm{i}s} Bf = e^{\theta_1 s} B|A|^{\mathrm{i}s} f,\ s \in \mathbb{R},$ for all $f \in \mathcal{D}_{q^2}(A^2, B)$. Since $\mathcal{D}_{q^2}(A^2, B)$ is a core for B by assumption, the latter implies that $|A|^{\mathrm{i}s} B \subseteq e^{\theta_1 s} B|A|^{\mathrm{i}s}$ for $s \in \mathbb{R}$. Therefore, since we also assumed that $\mathcal{D}_q(A, B)$ is a core for B, the assumptions of Proposition 19 are fulfilled with $k = 1$, so the pair $\{A, B\}$ belongs to the class $\mathcal{C}_1$. □

Related characterizations of the class $\mathcal{C}_1$ can be also given by requiring the weak resolvent relation for $|A|$ rather than A. The following theorem is a sample of such a result.

Theorem 24. *Suppose that* $0 < |\theta_0| < \pi$ *and* $\ker A = \ker B = \{0\}$. *Suppose that the domains* $\mathcal{D}_q(A, B)$ *and* $\mathcal{D}_{-q}(|A|, B)$ *are cores for* B *and*

$$-qR_{\lambda q}(|A|)B \subseteq BR_{-\lambda}(|A|) \quad \text{for} \quad \lambda > 0.$$

Then the pair $\{A, B\}$ *is in* $\mathcal{C}_1$.

Proof. The assumptions of Theorem 24 imply that the pair $\{|A|, B\}$ satisfies the assumptions of Theorem 21 for the relation (2) with $q = e^{-\mathrm{i}\theta_0}$ replaced by $-q = e^{-\mathrm{i}\theta_1}$. Note that $0 < |\theta_1| < \pi$, since we assumed that $0 < |\theta_0| < \pi$. Therefore, by Theorem 21, we have $|A|^{\mathrm{i}t} B \subseteq e^{\theta_1 t} B|A|^{\mathrm{i}t}$ for all $t \in \mathbb{R}$. Since $\mathcal{D}_q(A, B)$ is a core for B, it follows from Proposition 19 that the pair $\{A, B\}$ is in $\mathcal{C}_1$. □

The crucial assumption in the preceding Theorems 21, 22, and 23 is that the weak A-resolvent identity (9) holds on some line that intersects the critical sector only at the origin. In addition there have been technical assumptions such as $\ker A = \{0\}$ and the requirement that $\mathcal{D}_q(A, B)$ resp. $\mathcal{D}_{q^2}(A^2, B)$ is a core for the operator B. These technical assumptions can be avoided if we assume in addition that the weak B-resolvent identity (10) holds for some points of the resolvent set $\rho(B)$.

Theorem 25. *Let* A *is a positive operator. Suppose that* $0 < |\theta_0| < \pi$ *and there exist a number* $\nu \in \rho(B)$ *such that* $\nu\overline{q}, \overline{\nu}\,\overline{q} \in \rho(B)$,

$$\overline{q}\, R_{\nu\overline{q}}(B)A \subseteq AR_\nu(B) \quad \text{and} \quad \overline{q}\, R_{\overline{\nu}\,\overline{q}}(B)A \subseteq AR_{\overline{\nu}}(B). \tag{45}$$

Assume that

$$qR_{-\lambda q}(A)B \subseteq BR_{-\lambda}(A) \quad \text{for all} \quad \lambda > 0. \tag{46}$$

Then the pair $\{A, B\}$ is an orthogonal direct sum of trivial representations $\{A_2, 0\}$ and $\{0, B_2\}$ and a pair $\{A_0, B_0\}$ of the class $\mathcal{C}_0$.

Proof. From (45) it follows that the operator $R_\nu(B)$ and its adjoint $R_{\overline{\nu}}(B)$ leave the closed subspace $\mathcal{G}_2 := \ker A$ invariant. Therefore, $\mathcal{G}_2$ is reducing for $R_\nu(B)$ and hence for B. Since $\mathcal{G}_2 \equiv \ker A$ is obviously reducing for A, the pair $\{A, B\}$ decomposes as an orthogonal sum of pairs $\{0, B_2\}$ and $\{\tilde{A}, \tilde{B}\}$ such that $\ker \tilde{A} = \{0\}$. Further, by Proposition 5(vi), (45) implies that $\mathcal{D}_q(A, B)$ is core for B. Hence $\mathcal{D}_q(\tilde{A}, \tilde{B})$ is core for $\tilde{B}$. Clearly, (46) leads to the relation $R_{-\lambda q}(\tilde{A})\tilde{B} \subseteq \tilde{B}R_{-\lambda}(\tilde{A})$ for $\lambda > 0$. Thus, the pair $\{\tilde{A}, \tilde{B}\}$ satisfies the assumptions of Theorem 21 which gives the assertion. □

Theorem 26. *Suppose that $0 < |\theta_0| < \frac{\pi}{2}$ and there exists a number $\nu \in \rho(B)$ such that $\nu\overline{q}$, $\overline{\nu}\,\overline{q}$, $\nu\,\overline{q}^2 \in \rho(B)$ and*

$$\overline{q}\, R_{\lambda\overline{q}}(B)A \subseteq AR_\lambda(B) \quad \text{for} \quad \lambda = \nu\overline{q}\,,\ \nu,\ \overline{\nu}. \tag{47}$$

Assume that

$$qR_{\mu \mathrm{i} q}(A)B \subseteq BR_{\mu\mathrm{i}}(A) \ \text{ for all } \ \mu \in \mathbb{R}, \mu \neq 0.$$

Then $\{A, B\}$ is an orthogonal sum of trivial representations and a pair $\{A_0, B_0\}$ belonging to the class $\mathcal{C}_0$.

Proof. Arguing in a similar manner as in the proof of Theorem 25 the assertion is reduced to Theorem 22. We sketch only the necessary modifications: From the relations (47), applied for $\lambda = \nu, \overline{\nu}$, it follows that $\ker A$ is reducing for the pair $\{A, B\}$. By Proposition 5(vi) and Corollary 7, the relations (47), applied with $\lambda = \nu, \nu\overline{q}$, imply that the domains $D_q(A, B)$ and $D_{q^2}(A^2, B)$ are cores for B. □

Theorem 27. *Suppose that $\frac{\pi}{2} < |\theta_0| < \pi$ and there exist numbers $\nu \in \rho(B)$ and $p \in \mathbb{C}\backslash\mathcal{S}(-q)$ such that $\nu\overline{q}$, $\overline{\nu}\,\overline{q}$, $\nu\,\overline{q}^2 \in \rho(B)$,*

$$\overline{q}\, R_{\lambda\overline{q}}(B)A \subseteq AR_\lambda(B) \quad \text{for} \quad \lambda = \nu\overline{q}\,,\ \nu,\ \overline{\nu}, \tag{48}$$

$$qR_{\overline{p}\,q}(A)B \subseteq BR_{\overline{p}}(A), \tag{49}$$

$$qR_{\mu pq}(A)B \subseteq BR_{\mu p}(A) \quad \text{for all} \quad \mu \in \mathbb{R}, \mu \neq 0. \tag{50}$$

Then $\{A, B\}$ is an orthogonal sum of trivial representations and a pair $\{A_1, B_1\}$ of the class $\mathcal{C}_1$.

Proof. As in the proof of Theorem 26 it follows from (48) that $\ker A$ is reducing for the pair $\{A, B\}$ and that $D_q(A, B)$ and $D_{q^2}(A^2, B)$ are cores for B. Likewise the relations $qR_{\overline{p}\,q}(A)B \subseteq BR_{\overline{p}}(A)$ and $qR_{pq}(A)B \subseteq BR_p(A)$ (by (49) and (50)) imply that $\ker B$ is reducing for the pair $\{A, B\}$. Using these facts the assertion is derived from Theorem 23. □

Remarks. By Theorem 20, the weak B-resolvent relation $\overline{q}R_{\mu\overline{q}}(B)A \subseteq AR_\mu(B)$ is satisfied for a pair $\{A, B\}$ of the class $\mathcal{C}_0$ if $\mu \in \mathbb{C}\backslash\mathcal{S}(\overline{q})$ and for a pair $\{A, B\}$ in $\mathcal{C}_1$ if $\mu \in \mathbb{C}\backslash\mathcal{S}(-\overline{q})$. From this result it follows that for the corresponding pairs in Theorems 25–27 the assumptions (45), (47), and (48) can be fulfilled. That is, if $\{A, B\}$ is a pair of the class $\mathcal{C}_0$ we can choose $\nu \in \mathbb{C}\backslash\mathcal{S}(\overline{q})$ such that $\overline{\nu} \in \mathbb{C}\backslash\mathcal{S}(\overline{q})$ (then (45) holds) and if in addition $0 < |\theta_0| < \frac{\pi}{2}$ there exists $\nu \in \mathbb{C}\backslash\mathcal{S}(\overline{q})$ such that $\nu\overline{q}$, $\overline{\nu} \in \mathbb{C}\backslash\mathcal{S}(\overline{q})$ (which implies (47)). If the pair $\{A, B\}$ is in $\mathcal{C}_1$ and $\frac{\pi}{2} < |\theta_0| < \pi$ we can find $\nu \in \mathbb{C}\backslash\mathcal{S}(-\overline{q})$ such that $\nu\overline{q}\,, \overline{\nu} \in \mathbb{C}\backslash\mathcal{S}(-\overline{q})$ (these conditions imply (48))

and there exists $p \in \mathbb{C}\backslash\mathcal{S}(-q)$ such that $\overline{p} \in \mathbb{C}\backslash\mathcal{S}(-q)$ (then (46) and (50) are satisfied by Theorem 20(ii)).

6. Deficiency subspaces and their dimensions

Let A and B be positive self-adjoint operators with trivial kernels acting on a Hilbert space $\mathcal{H}$. Then, by Theorems 20 and 21 and by Proposition 5, the pair $\{A, B\}$ belongs to the class $\mathcal{C}_0$ of representations of equation (2) if and only if the resolvent relations (12) and (13) are satisfied for all $\lambda, \mu \in \mathbb{C}$ such that $\lambda, \mu \notin \mathcal{S}(q)^+$. Moreover, up to unitary equivalence, the only irreducible such pair is $\{e^{\alpha Q}, e^{\beta P}\}$, where $|\alpha\beta| < \pi$ and $q = e^{-\mathrm{i}\alpha\beta}$; see Section 3. In this section we study the resolvent relations for the pair $\{e^{\alpha Q}, e^{\beta P}\}$ in the general case, that is, we do not assume that $|\alpha\beta| < \pi$ or $\lambda, \mu \notin \mathcal{S}(q)^+$.

First let us fix some assumptions and notations that will be kept throughout this section. Suppose that $\alpha, \beta \in \mathbb{R}$ and $q := e^{-\mathrm{i}\alpha\beta} \neq \pm 1$. We consider the pair

$$\{A := e^{\alpha Q}, B := e^{\beta P}\}$$

of self-adjoint operators on the Hilbert space $\mathcal{H} := L^2(\mathbb{R})$. Recall that A and B act by

$$(Af)(x) = e^{\alpha x} f(x) \quad \text{and} \quad (Bf)(x) = f(x + \mathrm{i}\beta) \tag{51}$$

for all functions f of the dense domain

$$\mathcal{D}_0 = \mathrm{Lin}\,\{e^{-\varepsilon x^2 + \gamma x}; \varepsilon > 0, \gamma \in \mathbb{C}\}.$$

Since (2) is satisfied for $f \in \mathcal{D}_0$ (by (51)), we know from Section 2 that (4) holds for all $h \in (B - \mu I)(A - \lambda I)\mathcal{D}_0$ and that (7) holds for all $h \in (A - \lambda q I)(B - \mu q I)\mathcal{D}_0$. Therefore, in order to find the resolvent equations in the present case it suffices to describe the form of resolvent equations on the orthogonal complements of spaces $(B - \mu I)(A - \lambda I)\mathcal{D}_0$ and $(A - \lambda q I)(B - \mu q I)\mathcal{D}_0$, respectively. This will be achieved by the formulas at the end of this section.

Definition 4.

$$\mathcal{H}_A(\lambda, \mu) = \{\psi \in \mathcal{H} : \psi \perp (B - \mu I)(A - \lambda I)\mathcal{D}_0\},$$
$$\mathcal{H}_B(\lambda, \mu) = \{\eta \in \mathcal{H} : \eta \perp (A - \lambda q I)(B - \mu q I)\mathcal{D}_0\}.$$

Assume that $\lambda,\ \mu \in \mathbb{C} \setminus (\mathbb{R}_+ \cup \bar{q}\mathbb{R}_+)$, where $\mathbb{R}_+ = [0, +\infty)$, and write

$$\lambda = e^{r + \mathrm{i}s},\ \mu = e^{u + \mathrm{i}v}, \quad r > 0,\ u > 0,\ |s| < \pi,\ |v| < \pi.$$

Further, we let

$$\alpha\beta = \theta_0 + 2\pi m, \quad m \in \mathbb{Z},\ \theta_0 \in (-\pi, \pi),$$

so that $q = e^{-\mathrm{i}\theta_0}$, and set

$$\varepsilon_1 = \mathrm{sign}(v),\ \varepsilon_2 = \mathrm{sign}(v - \theta_0),\ \varepsilon_3 = \mathrm{sign}(s),\ \varepsilon_4 = \mathrm{sign}(s - \theta_0).$$

Theorem 28. (i) *The vector space $\mathcal{H}_A(\lambda, \mu)$ is spanned by the functions*

$$\psi_j(x) = \frac{\bar{\mu}^{-\mathrm{i}x/\beta} e^{\pi(1+\varepsilon_1)x/\beta}}{e^{2\pi x/\beta} - \bar{\lambda}^{2\pi/\alpha\beta} e^{-4\mathrm{i}\pi^2 j/\alpha\beta}},$$

where $j \in \mathbb{Z}$ and $0 < (s - 2\pi j)/\alpha\beta < 1$, and its dimension is

$$\dim \mathcal{H}_A(\lambda, \mu) = \mathrm{sign}(\alpha\beta)\Big(m + \frac{\varepsilon_3 - \varepsilon_4}{2}\Big).$$

(ii) *The space* $\mathcal{H}_B(\lambda,\mu)$ *is the linear span of functions*

$$\eta_k(x) = \frac{(\bar{\mu}\bar{q})^{-\mathrm{i}x/\beta} e^{2\pi kx/\beta}}{e^{\alpha x} - \bar{\lambda}\bar{q}}.$$

where $k \in \mathbb{Z}$ *and* $0 < (\theta_0 + 2\pi k - v)/\alpha\beta < 1$*, and it has the dimension*

$$\dim \mathcal{H}_B(\lambda,\mu) = \operatorname{sign}(\alpha\beta)\Big(m + \frac{\varepsilon_1 - \varepsilon_2}{2}\Big).$$

Proof. (i): First we study the space $\mathcal{H}_B(\lambda,\mu)$. Let $\eta \in \mathcal{H}_B(\lambda,\mu)$. Applying (51) to $\phi(x) = e^{-\varepsilon x^2 + itx} \in \mathcal{D}_0$ we compute

$$\begin{aligned}((A-\lambda q)(B-\mu q)\phi)(x) &= (e^{\alpha x} - \lambda q)(e^{-\varepsilon(x+i\beta)^2 + it(x+i\beta)} - \mu q e^{-\varepsilon x^2 + itx}) \\ &= (e^{\alpha x} - \lambda q)(e^{-2i\beta\varepsilon x + \varepsilon\beta^2 - t\beta} - \mu q) e^{-\varepsilon x^2 + itx}.\end{aligned}$$

Therefore, since $\eta \perp (A-\lambda q)(B-\mu q)\phi(x)$, we obtain

$$\begin{aligned}0 &= \int_{\mathbb{R}} \overline{\eta(x)}\,(e^{\alpha x} - \lambda q)(e^{-2i\beta\varepsilon x + \varepsilon\beta^2 - t\beta} - \mu q) e^{-\varepsilon x^2 + itx}\,dx \\ &= e^{\varepsilon\beta^2 - t\beta} \int_{\mathbb{R}} e^{i(t-2\beta\varepsilon)x}\,\overline{\eta(x)}\,(e^{\alpha x} - \lambda q)\,e^{-\varepsilon x^2}\,dx \\ &\quad - \mu q \int_{\mathbb{R}} e^{itx}\,\overline{\eta(x)}\,(e^{\alpha x} - \lambda q)\,e^{-\varepsilon x^2}\,dx\end{aligned}$$

which implies that

$$e^{\varepsilon\beta^2 - t\beta} g_\varepsilon(t - 2\beta\varepsilon) = \mu q\, g_\varepsilon(t), \qquad g_\varepsilon(t) = \int_{\mathbb{R}} e^{itx}\,\overline{\eta(x)}\,(e^{\alpha x} - \lambda q)\,e^{-\varepsilon x^2}\,dx. \tag{52}$$

Since $\mu = e^{u+iv}$, $|v| < \pi$, it is easily seen that the function

$$G_\varepsilon(t) = e^{-\frac{t^2}{4\varepsilon} - \frac{u + i(v-\theta_0)}{2\varepsilon\beta} t}$$

satisfies (52). Therefore, any solution of (52) has the form $G_\varepsilon(t)H_\varepsilon(t)$, where H_ε is a periodic function on $\mathbb{R}$ with period $-2\varepsilon\beta$, that is, $H_\varepsilon(t - 2\varepsilon\beta) = H_\varepsilon(t)$ for $t \in \mathbb{R}$.

The crucial step of this proof is contained in the following lemma.

Lemma 29. H_ε *is a trigonometric polynomial, that is, there are an integer* $l \geq 0$ *and numbers* $c_k \in \mathbb{C}$*,* $k = -l, \dots, l$ *such that* $H_\varepsilon(t) = \sum_{k=-l}^{l} d_k e^{\frac{\mathrm{i}\pi kt}{\varepsilon\beta}}$.

Proof. We have

$$H_\varepsilon(t) = (G_\varepsilon(t))^{-1} g_\varepsilon(t) = e^{\frac{t^2}{4\varepsilon} + \frac{u + i(v-\theta_0)}{2\varepsilon\beta} t} \int_{\mathbb{R}} e^{itx}\,\overline{\eta(x)}\,(e^{\alpha x} - \lambda q)\,e^{-\varepsilon x^2}\,dx. \tag{53}$$

Because the above arguments are valid for complex numbers t as well, H_ε becomes a periodic function on a whole complex plane $\mathbb{C}$.

Since $\eta \in L_2(\mathbb{R})$, the function $e^{\tau|x|}\overline{\eta(x)}\,(e^{\alpha x} - \lambda q)\,e^{-\varepsilon x^2}$ is also in $L_2(\mathbb{R})$ for any $\tau > 0$. Therefore, the integral in (53) is an entire function on the complex plane $\mathbb{C}$.

We show that H_ε is of exponential type. For any $s \in \mathbb{R}$ we have

$$\begin{aligned}|H_\varepsilon(t + \mathrm{i}s)| &= |e^{\frac{(t+\mathrm{i}s)^2}{4\varepsilon} + \frac{u + \mathrm{i}(v-\theta_0)}{2\varepsilon\beta}(t+\mathrm{i}s)}|\Big|\int_{\mathbb{R}} e^{\mathrm{i}(t+\mathrm{i}s)x}\,\overline{\eta(x)}\,(e^{\alpha x} - \lambda q)\,e^{-\varepsilon x^2}\,dx\Big| \\ &= e^{\frac{t^2 - s^2}{4\varepsilon} + \frac{ut - (v-\theta_0)s}{2\varepsilon\beta}} \Big|\int_{\mathbb{R}} e^{\mathrm{i}tx}\,\overline{\eta(x)}\,(e^{\alpha x} - \lambda q)\,e^{-\varepsilon x^2 - sx}\,dx\Big| \\ &\leq e^{\frac{t^2 - s^2}{4\varepsilon} + \frac{ut - (v-\theta_0)s}{2\varepsilon\beta}} \|\eta\| \Big(\int_{\mathbb{R}} |e^{\alpha x} - \lambda q|^2\,e^{-2\varepsilon x^2 - 2sx}\,dx\Big)^{1/2}.\end{aligned}$$

Since

$$\int_{\mathbb{R}} |e^{\alpha x} - \lambda q|^2 \, e^{-2\varepsilon x^2 - 2sx} \, dx = \sqrt{\frac{\pi}{2\varepsilon}} e^{\frac{s^2}{2\varepsilon}} \big(|\lambda|^2 + e^{\frac{\alpha(\alpha-2s)}{2\varepsilon}} - (\lambda q + \bar{\lambda}\bar{q}) e^{\frac{\alpha(\alpha-4s)}{8\varepsilon}}\big),$$

for $0 \le t \le 2\varepsilon\beta$ we have

$$|H_\varepsilon(t + \mathrm{i}s)| \le M e^{-\frac{(v-\theta_0)s}{2\varepsilon\beta}} \|\eta\| \Big(\sqrt{\frac{\pi}{2\varepsilon}} \big(|\lambda|^2 + e^{\frac{\alpha(\alpha-2s)}{2\varepsilon}} - (\lambda q + \bar{\lambda}\bar{q}) e^{\frac{\alpha(\alpha-4s)}{8\varepsilon}}\big) \Big)^{1/2},$$

where M is the supremum of $e^{\frac{t^2}{4\varepsilon} + \frac{ut}{2\varepsilon\beta}}$ on the interval $[0, 2\varepsilon\beta]$. Since $H_\varepsilon(t + \mathrm{i}s)$ is periodic in t, the latter estimate holds for all $t \in \mathbb{R}$. Hence $H_\varepsilon(t + \mathrm{i}s)$ has exponential growth. Thus, H_ε is an entire periodic function of exponential type. By a result from complex analysis (see, e.g., [8, p. 334]), such a function H_ε has to be a trigonometric polynomial

$$H_\varepsilon(t) = \sum_{k=-l}^{l} c_k e^{\frac{\mathrm{i}\pi k t}{\beta\varepsilon}}. \qquad \square$$

By Lemma 29 the equality $g_\varepsilon(t) = G_\varepsilon(t) H_\varepsilon(t)$ takes the form

$$\int_{\mathbb{R}} e^{itx} \, \overline{\eta(x)} \, (e^{\alpha x} - \lambda q) \, e^{-\varepsilon x^2} \, dx = e^{-\frac{t^2}{4\varepsilon} - \frac{u + i(v-\theta_0)}{2\varepsilon\beta} t} \sum_{k=-l}^{l} c_k e^{\frac{\mathrm{i}\pi k t}{\beta\varepsilon}}.$$

Applying the Fourier transform, we obtain

$$\begin{aligned} \overline{\eta(x)} \, (e^{\alpha x} - \lambda q) \, e^{-\varepsilon x^2} &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\mathrm{i}tx} g_\varepsilon(t) \, dt \\ &= e^{-\varepsilon x^2} e^{\mathrm{i}xu/\beta} e^{-(v-\theta_0)x/\beta} \sum_{k=-l}^{l} d_k e^{2\pi k x/\beta}, \end{aligned}$$

where

$$d_k = \frac{\sqrt{\varepsilon}}{\sqrt{\pi}} c_k e^{(u + \mathrm{i}(v - \theta_0 - 2k\pi))^2/(4\varepsilon\beta^2)}.$$

Obviously, the factor $e^{-\varepsilon x^2}$ cancels, so we get

$$\overline{\eta(x)} = \frac{e^{\mathrm{i}xu/\beta} e^{-(v-\theta_0)x/\beta}}{e^{\alpha x} - \lambda q} \sum_{k=-l}^{l} d_k e^{\frac{2\pi k x}{\beta}} = \frac{(\mu q)^{\mathrm{i}x/\beta}}{e^{\alpha x} - \lambda q} \sum_{k=-l}^{l} d_k e^{2\pi k x/\beta}. \tag{54}$$

Let us introduce the functions

$$\eta_k(x) := \frac{(\bar{\mu}\bar{q})^{-\mathrm{i}x/\beta} e^{2\pi k x/\beta}}{e^{\alpha x} - \bar{\lambda}\bar{q}} = \frac{e^{-\mathrm{i}ux/\beta} e^{(\theta_0 - v + 2\pi k)x/\beta}}{e^{\alpha x} - \bar{\lambda}\bar{q}}.$$

Then we have $\eta(x) = \sum_{k=-l}^{l} \bar{d}_k \eta_k(x)$.

Next we want to decide which functions η_k belong to $L_2(\mathbb{R})$. Let us begin with the case where $\alpha > 0$. Then the function $\eta_k(x)$ behaves like $e^{(\theta_0 - v + 2\pi k)x/\beta}$ as $x \to -\infty$ and like $e^{(\theta_0 - v + 2\pi k - \alpha\beta)x/\beta}$ as $x \to +\infty$, so that $\eta_k \in L^2(\mathbb{R})$ if and only $0 < (\theta_0 - v + 2\pi k)/\beta < \alpha$. In the case where $\alpha < 0$ a similar reasoning shows that $\eta_k \in L_2(\mathbb{R})$ if and only $\alpha < (\theta_0 - v + 2\pi k)/\beta < 0$. Thus, in both cases we have $\eta_k \in L^2(\mathbb{R})$ if and only if if and only if $0 < (\theta_0 - v + 2\pi k)/\alpha\beta < 1$, that is,

$$\begin{aligned} v - \theta_0 < 2\pi k < v + 2\pi m \quad &\text{for} \quad \alpha\beta > 0, \\ v + 2\pi m < 2\pi k < v - \theta_0 \quad &\text{for} \quad \alpha\beta < 0, \end{aligned}$$

or equivalently,

$$\frac{1+\varepsilon_2}{2} \le k \le m - \frac{1-\varepsilon_1}{2} \quad \text{for } \alpha\beta > 0, \quad m + \frac{1+\varepsilon_1}{2} \le k \le \frac{\varepsilon_2 - 1}{2} \quad \text{for } \alpha\beta < 0.$$

The functions η_k are obviously linearly independent. From the asymptotic behaviour of η_k it follows that $\eta(x) = \sum_{k=-l}^{l} \bar{d}_k \eta_k(x)$ is in $L^2(\mathbb{R})$ if and only each η_k with nonvanishing coefficient $\bar{d}_k$ is in $L^2(\mathbb{R})$. Therefore, we obtain

$$\dim \mathcal{H}_B = \operatorname{sign}(\alpha\beta)\Big(m + \frac{\varepsilon_1 - \varepsilon_2}{2}\Big).$$

(ii): Now we turn to the space $\mathcal{H}_A(\lambda, \mu)$. Let $\psi \in L^2(\mathbb{R})$. Recall that $\psi \in \mathcal{H}_A(\lambda, \mu)$ if and only if

$$\psi \perp (B - \mu I)(A - \lambda I)\mathcal{D}_0 = (e^{\beta P} - \mu I)(e^{\alpha Q} - \lambda I)\mathcal{D}_0$$

For the Fourier transform $\mathcal{F}$ we have $e^{\alpha Q} = \mathcal{F} e^{-\alpha P} \mathcal{F}^{-1}$ and $e^{\beta P} = \mathcal{F} e^{\beta Q} \mathcal{F}^{-1}$. Hence the latter is equivalent to

$$\psi \perp \mathcal{F}(e^{\beta Q} - \mu I)(e^{-\alpha P} - \lambda I)\mathcal{F}^* \mathcal{D}_0.$$

Since $\mathcal{D}_0$ is invariant under the Fourier transform, $\psi \in \mathcal{H}_A(\lambda, \mu)$ if and only if the inverse Fourier transform $\mathcal{F}^*\psi$ of ψ satisfies

$$\mathcal{F}^*\psi \perp (e^{\beta Q} - \mu I)(e^{-\alpha P} - \lambda I)\mathcal{D}_0.$$

From the proof of (i) we already know that $\psi \in \mathcal{H}_A(\lambda, \mu)$ if and only if $\mathcal{F}^*\psi$ is a linear combination of functions

$$\phi_k(x) = \frac{(\bar{\lambda})^{\mathrm{i}x/\alpha} e^{-2\pi k x/\alpha}}{e^{\beta x} - \bar{\mu}},$$

where $k \in \mathbb{Z}$ and $0 < (s - 2\pi k)/\alpha\beta < 1$. The condition on k can be rewritten as

$$\begin{aligned} \frac{1+\varepsilon_4}{2} - m \le k \le \frac{\varepsilon_3 - 1}{2} \quad &\text{for} \quad \alpha\beta > 0, \\ \frac{1+\varepsilon_3}{2} \le k \le \frac{\varepsilon_4 - 1}{2} - m \quad &\text{for} \quad \alpha\beta < 0. \end{aligned}$$

Therefore, we have

$$\dim \mathcal{H}_A = \operatorname{sign}(\alpha\beta)\Big(m + \frac{\varepsilon_3 - \varepsilon_4}{2}\Big).$$

To calculate the Fourier transform of ϕ_k, we shall apply the following formula (see, e.g., [4, 3.311.9])

$$\int_{\mathbb{R}} \frac{e^{-\delta x}}{e^{-x} + \gamma}\, dx = \frac{\pi \gamma^{\delta - 1}}{\sin \pi\delta}, \qquad |\arg \gamma| < \pi,\ 0 < \operatorname{Re}\delta < 1. \tag{55}$$

After some computations using (55) we obtain $\mathcal{F}\phi_k(x) = C_k \psi_k(x)$, where

$$\begin{aligned} C_k &= \frac{\mathrm{i}\sqrt{2\pi} e^{\mathrm{i}(u - \mathrm{i}v - \mathrm{i}\pi(1-\varepsilon_1))(r - \mathrm{i}s + \mathrm{i}2\pi k)/\alpha\beta}}{\bar{\mu}\beta}, \\ \psi_k(x) &= \frac{\bar{\mu}^{-\mathrm{i}t/\beta} e^{\pi(1+\varepsilon_1)t/\beta}}{(e^{2\pi t/\beta} - \bar{\lambda}^{2\pi/\alpha\beta} e^{4\mathrm{i}\pi^2 k/\alpha\beta})}. \end{aligned}$$

□

The following corollary restates the result on the dimensions of Theorem 28 in an important special case.

Corollary 30. *Suppose that $\{A, B\}$ is an irreducible nontrivial representation of the class $\mathcal{C}_0$ for relation (2). Then*

$$\dim \mathcal{H}_A(\lambda,\mu) = 0 \quad \text{for} \quad \lambda \notin \mathcal{S}(q)^+, \quad \dim \mathcal{H}_A(\lambda,\mu) = 1 \quad \text{for} \quad \lambda \in \mathcal{S}(q)^+,$$
$$\dim \mathcal{H}_B(\lambda,\mu) = 0 \quad \text{for} \quad \mu \notin \mathcal{S}(q)^+, \quad \dim \mathcal{H}_B(\lambda,\mu) = 1 \quad \text{for} \quad \mu \in \mathcal{S}(q)^+.$$

Proof. By Corollary 16, $\{A, B\}$ is unitarily equivalent to a pair $\{\delta_1 e^{\alpha P}, \delta_2 e^{\beta Q}\}$, where $\delta_1, \delta_2 \in \{1, -1\}$ and $\alpha\beta = \theta_0$, $|\theta_0| < \pi$. Hence the assertion follows at once from Theorem 28. □

We now continue the considerations towards the modified resolvent relations.

Proposition 31. *(i)* $R_{\bar\mu}(B)R_{\bar\lambda}(A) - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, I$ *maps* $\mathcal{H}_B(\lambda,\mu)$ *into* $\mathcal{H}_A(\lambda,\mu)$.
(ii) $R_{\bar\lambda\bar q}(A)R_{\bar\mu\bar q}(B) + \frac{1}{\bar q\bar\lambda\bar\mu(1-\bar q)}\, I$ *maps* $\mathcal{H}_A(\lambda,\mu)$ *into* $\mathcal{H}_B(\lambda,\mu)$.

Proof. (i): Indeed, for $\phi \in \mathcal{D}_0$ we have

$$(B - \mu I)(A - \lambda I)\phi = \bar q(A - \lambda q I)(B - \mu q I)\phi + \lambda\mu(1-q)\phi.$$

Therefore, for $\eta \in \mathcal{H}_B(\lambda,\mu)$ we derive

$$\begin{aligned}
&\Big\langle \big(R_{\bar\mu}(B)R_{\bar\lambda}(A) - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, I\big)\eta, (B-\mu I)(A-\lambda I)\phi \Big\rangle \\
&= \langle R_{\bar\mu}(B)R_{\bar\lambda}(A)\eta, (B-\mu I)(A-\lambda I)\phi\rangle - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, \langle \eta, (B-\mu I)(A-\lambda I)\phi\rangle \\
&= -\frac{q}{\bar\lambda\bar\mu(1-\bar q)}\, \langle \eta, (A - \lambda q I)(B - \mu q I)\phi\rangle = 0,
\end{aligned}$$

that is, $\big(R_{\bar\mu}(B)R_{\bar\lambda}(A) - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, I\big)\eta \in \mathcal{H}_A(\lambda,\mu)$.
(ii) is proved in a similar manner. □

From Proposition 31 it follows that the operator

$$\Big(R_{\bar\mu}(B)R_{\bar\lambda}(A) - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, I\Big)\Big(R_{\bar\lambda\bar q}(A)R_{\bar\mu\bar q}(B) + \frac{1}{\bar q\bar\lambda\bar\mu(1-\bar q)}\, I\Big) \tag{56}$$

maps the subspace $\mathcal{H}_A(\lambda,\mu)$ into itself and that the operator

$$\Big(R_{\bar\lambda\bar q}(A)R_{\bar\mu\bar q}(B) + \frac{1}{\bar q\bar\lambda\bar\mu(1-\bar q)}\, I\Big)\Big(R_{\bar\mu}(B)R_{\bar\lambda}(A) - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, I\Big) \tag{57}$$

maps $\mathcal{H}_B(\lambda,\mu)$ into itself. In the special case when $\mathcal{H}_A(\lambda,\mu) = \mathcal{H}_B(\lambda,\mu) = \{0\}$ we know from Section 2 that the corresponding operators in (56) and (57) are both equal to $-q\bar\lambda^{-2}\bar\mu^{-2}(1-\bar q)^{-2}\, I$ on the whole Hilbert space.

In the general case some lengthy but straightforward computations using (55) and the relations $e^{\alpha Q} = \mathcal{F} e^{-\alpha P} \mathcal{F}^{-1}$, $e^{\beta P} = \mathcal{F} e^{\beta Q} \mathcal{F}^{-1}$, lead to the formulas

$$\Big(R_{\bar\lambda\bar q}(A) R_{\bar\mu\bar q}(B) + \frac{1}{\bar\lambda\bar\mu\bar q(1-\bar q)}\, I\Big)\psi_j(x)$$
$$= \frac{\bar\lambda^{2\pi(m-(1-\varepsilon_1)/2)/\alpha\beta} e^{-4i\pi^2(m-(1-\varepsilon_1)/2)j/\alpha\beta}}{\bar\lambda\bar q(1-\bar q)\bar\mu} \sum_{l=(1+\varepsilon_2)/2}^{m-(1-\varepsilon_1)/2} \bar\lambda^{-2\pi l/\alpha\beta} e^{4i\pi^2 lj/\alpha\beta} \eta_l(x),$$
$$\Big(R_{\bar\mu}(B) R_{\bar\lambda}(A) - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, I\Big)\eta_k(x)$$
$$= \frac{-2\pi\bar\lambda^{-2\pi(m-k+(\varepsilon_1-1)/2)/\alpha\beta}}{\alpha\beta\bar\lambda\bar\mu(1-\bar q)} \sum_{j=(1-\varepsilon_3)/2}^{m-(\varepsilon_4+1)/2} e^{4i\pi^2 j(m-k+(\varepsilon_1-1)/2)/\alpha\beta} \psi_j(x).$$

Therefore, we finally obtain

$$\Big(R_{\bar\mu}(B) R_{\bar\lambda}(A) - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, I\Big)\Big(R_{\bar\lambda\bar q}(A) R_{\bar\mu\bar q}(B) + \frac{1}{\bar\lambda\bar\mu\bar q(1-\bar q)}\, I\Big)\psi_j(x)$$
$$= \frac{-2\pi}{\alpha\beta\bar\lambda^2\bar\mu^2\bar q(1-\bar q)^2} \sum_{k=0}^{m-1-(\varepsilon_2-\varepsilon_1)/2} \sum_{l=(1-\varepsilon_3)/2}^{m-(\varepsilon_4+1)/2} e^{-4i\pi^2 kj/\alpha\beta} e^{4i\pi^2 lk/\alpha\beta} \psi_l(x),$$
$$\Big(R_{\bar\lambda\bar q}(A) R_{\bar\mu\bar q}(B) + \frac{1}{\bar\lambda\bar\mu\bar q(1-\bar q)}\, I\Big)\Big(R_{\bar\mu}(B) R_{\bar\lambda}(A) - \frac{1}{\bar\lambda\bar\mu(1-\bar q)}\, I\Big)\eta_k$$
$$= \frac{-2\pi}{\alpha\beta\bar\lambda^2\bar\mu^2\bar q(1-\bar q)^2} \sum_{j=(1-\varepsilon_3)/2}^{m-(\varepsilon_4+1)/2} \sum_{l=(1+\varepsilon_2)/2}^{m-(1-\varepsilon_1)/2} e^{-4i\pi^2 j(k-l)/\alpha\beta} \bar\lambda^{2\pi(k-l)/\alpha\beta} \eta_l(x).$$

The preceding two equations are the versions of the resolvent equations for basis elements of the subspaces $\mathcal{H}_A(\lambda,\mu)$ and $\mathcal{H}_B(\lambda,\mu)$, respectively.

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Institute of Mathematics, Ukrainian Academy of Sciences, Tereshchenkivska 3, Kyiv 01601, Ukraine

E-mail address: `vo@imath.kiev.ua`

Mathematisches Institut, Universität Leipzig, Augustusplatz 9/10, 04109 Leipzig, Germany

E-mail address: `schmuedgen@math.uni-leipzig.de`